

Harrison transformation and charged black objects in Kaluza-Klein theory

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ABSTRACT: We generate charged black brane solutions in D -dimensions in a theory of gravity coupled to a dilaton and an antisymmetric form, by using a Harrison-type transformation. The seed vacuum solutions that we use correspond to uplifted Kaluza-Klein black strings and black holes in $(D-p)$ -dimensions. A generalization of the Marolf-Mann quasilocal formalism to the Kaluza-Klein theory is also presented, the global charges of the black objects being computed in this way. We argue that the thermodynamics of the charged solutions can be derived from that of the vacuum configurations. Our results show that all charged Kaluza-Klein solutions constructed by means of Harrison transformations are thermodynamically unstable in a grand canonical ensemble. The general formalism is applied to the case of nonuniform black strings and caged black hole solutions in $D = 5, 6$ Einstein-Maxwell-dilaton gravity, whose geometrical properties and thermodynamics are discussed. We argue that the topology changing transition scenario, which was previously proposed in the vacuum case, also holds in this case. Spinning generalizations of the charged black strings are constructed in six dimensions in the slowly rotating limit. We find that the gyromagnetic ratio of these solutions possesses a nontrivial dependence on the nonuniformity parameter.

KEYWORDS: Black Strings, p-branes, Einstein-Maxwell-Dilaton gravity.

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1. Introduction

Finding new solutions of the theory of general relativity by using solution generating techniques is a subject of long standing interest. One of the main approaches consists in the use of hidden symmetries (dualities) arising in dimensionally reduced theories. This technique was initially used in four-dimensional general relativity coupled to a Maxwell field [1] and

was generalized in recent years to higher dimensional gravity coupled to various matter fields, a great variety of important solutions being generated in this way.

This approach is particularly relevant in the case of Kaluza-Klein (KK) solutions with a nontrivial dependence on the extra-dimension. The configurations of interest here are the static nonuniform black strings (NUBS) and the KK localized black holes (BHs) (thus we shall not consider KK solutions with bubbles [2], [3], [4] or multicenter black objects [5]). The numerical results in the literature reveal in this case the existence of a complicated phase structure, the caged BH and the NUBS branches merging at a topology changing transition [6]. Most of the results in the literature on KK solutions with a dependence of the extra-dimension concern the case of vacuum configurations. Even in this case, there are no explicit solutions that one could write down in closed analytical form (both BHs and NUBSs cannot be found by using a Weyl ansatz [2] since the number of abelian Killing vectors is too small for them to belong to a generalised Weyl class). However, one can analyse their properties by using a combination of analytical and numerical methods, which is enough for most purposes.

The more realistic case of localized BHs and NUBS solutions with matter fields is relatively scarcely explored. Since it is extremely difficult to solve the field equations for such configurations, it would be very helpful if one could generate new solutions from known ones on the basis of internal symmetries of the field equations. The main purpose of this work is to present such a construction for charged black brane configurations in a theory of gravity coupled to a dilaton and a $(p+1)$ -form and formulated in D -dimensions. These solutions are generated by using a Harrison-type transformation originally proposed in [7], applied to the known KK vacuum nonuniform black strings and black holes in $D-p$ dimensions. Different from other cases previously discussed in the literature [8], this approach produces solutions for an arbitrary coupling of the dilaton field to the antisymmetric form.

To analyze the thermodynamic properties of such non-uniform brane solutions one has to be able to compute their total action and define the various conserved quantities. For asymptotically flat spacetimes there are several ways to compute the conserved charges and action. One particularly interesting approach is the so-called counterterm method, which was inspired by the AdS/CFT conjecture in string theory. In this approach one supplements the action by including suitable boundary counterterms, which are functionals of curvature invariants of the induced metric on the boundary. By choosing appropriate counterterms that cancel the divergencies in the total action, one further defines a boundary stress-tensor by varying the on-shell action with respect to the boundary metric. Using the boundary stress-tensor one can then construct a conserved charge for any asymptotic Killing vector. However, unlike the AdS case, here one does not have a universal prescription for defining the boundary counterterms and their form seems to depend heavily on the boundary topology. A novel counterterm for asymptotically flat spaces has been proposed recently by Mann and Marolf [9]. In contrast to the previous counterterms, by its definition the Mann-Marolf (MM) counterterm assigns zero action to the Minkowski spacetime in any dimension, while the conserved quantities are constructed essentially using the electric part of the Weyl conformal tensor. In this work, we also propose a generalization of the MM counterterm to extended objects, like the p -brane spacetimes. These spaces are not asymp-

totically flat but ‘*transverse asymptotically flat*’. Leaving a more general analysis for future work, here we show that the total action and conserved charges for p -branes can be defined in terms of the electric part of the Weyl tensor and the final results are consistent with those previously derived (by other means) in the literature [10, 11, 12, 13]. As a check we also compute the action and conserved charges using different choices of counterterms for asymptotically flat spaces and we obtain similar results. One should note here that, unlike the case of the MM counterterm, these other counterterm choices are severely limited as in higher than eight dimensions and in certain coordinates they cannot regularize the action even for flat space [14].

Once one computes the conserved charges and total action, one is ready to perform the standard thermodynamic analysis of the generated solutions. In particular, we proved that the thermodynamic properties of the charged solutions generated using the Harrison transformation can be simply derived from those of the vacuum seed configurations. The neutral and charged variables are related in a simple way. Moreover, we find that the thermodynamics of the system is affected by the value of the dilaton coupling constant. Our results show that all $D = 5, 6$ charged KK solutions constructed in this way are thermodynamically unstable in a grand canonical ensemble. However, when considering instead a canonical ensemble, one finds stable solutions for large enough values of the electric charge.

The outline of the paper is as follows. We begin with a presentation of the general ansatz and we give the necessary details regarding the Harrison transformation in the next section. In Section 3, we consider the computation of the global charges of black branes, proposing a generalization of the MM quasilocal formalism to the case of the KK theory. In Section 4 we review the basic properties of the caged BHs and NUBS vacuum configurations of the KK theory that are used as seed solutions in the Harrison transformation. Section 5 contains a general discussion of the charged black branes in KK theory. A more detailed analysis is presented there for $D = 5, 6$ NUBS and caged BHs in Einstein-Maxwell-dilaton theory. The static, charged NUBSs generated in this way are used in Section 6 to construct $D = 6$ spinning configurations. The results there are valid in the limit of slow rotation. We give our conclusions and remarks in the final section. The Appendix A contains a brief discussion of some technical aspects involved in the numerical construction of the caged BHs and NUBS vacuum solutions, including an independent confirmation for the existence of $D = 6$ caged BHs. In Appendix B we present an exact solution describing slowly rotating, charged uniform black strings in EM theory in D dimensions, and compute the gyromagnetic ratio of these configurations.

2. General formalism

2.1 The action and field equations

We consider the following action

$$I = -\frac{1}{16\pi G} \int_M d^D x \sqrt{-g} \left(R - \frac{1}{2} (\nabla\phi)^2 - \frac{e^{-2a\phi}}{2(p+2)!} F_{p+2}^2 \right) - \frac{1}{8\pi G} \int_{\partial M} d^{D-1} x \sqrt{-h} K, \quad (2.1)$$

with ϕ a dilaton field and F_{p+2} a $(p+2)$ -form field strength with $F_{p+2} = d\mathcal{A}_{p+1}$, where \mathcal{A}_{p+1} is the corresponding $(p+1)$ -form gauge field. The free parameter a governs the strength of the coupling of the dilaton to the form field. h_{ij} is the induced metric on the boundary $\partial\mathcal{M}$ and K_{ij} is the extrinsic curvature of this boundary, with $K = K_{ij}h^{ij}$. For suitable values of D, p, a , the action (2.1) corresponds to truncations of various supergravity models and also of the bosonic part of the low energy action of type IIA and IIB string theory.

The corresponding field equations are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= \frac{1}{2}T_{\mu\nu}, \\ \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) &= -\frac{a}{(p+2)!}e^{-2a\phi}F_{p+2}^2, \\ \partial_\mu\left(e^{-2a\phi}\sqrt{-g}F_{p+2}^{\mu\nu_1\dots\nu_{p+1}}\right) &= 0, \end{aligned} \quad (2.2)$$

where the total stress-energy tensor has contributions from the dilaton and the $(p+2)$ -field strength:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2 + e^{-2a\phi}\frac{1}{(p+1)!}(F_{\mu\nu_1\dots\nu_{p+1}}F_\nu^{\nu_1\dots\nu_{p+1}} - \frac{1}{2(p+2)}g_{\mu\nu}F_{p+2}^2).$$

Although we shall consider in this paper only solutions with an ‘electric’ potential \mathcal{A}_{p+1} , magnetically charged solutions can be obtained by using the generalized ‘electric-magnetic duality’ by taking:

$$a\phi \rightarrow -a\phi, \quad p \rightarrow D - p - 4, \quad F_{p+2} \rightarrow \tilde{F}_{D-p-2} = e^{2a\phi} * F_{p+2}, \quad (2.3)$$

where $*$ stands for the Hodge dual, the geometry being not affected by this transformation.

2.2 The Harrison transformation

Following the method outlined in [7] (see also [15]), we suppose that the space-time has (at least) $p+1$ commuting Killing vectors, one of them being timelike. Then the D -dimensional line element can be written in the following form:

$$ds^2 = g_{ij}^{(0)}(x)dy^i dy^j + g_{\alpha\beta}^{(0)}(x)dx^\alpha dx^\beta, \quad (2.4)$$

where $i, j = 1, \dots, p+1$, while $\alpha, \beta = 1, \dots, D-(p+1)$; also note that $g_0 = \det(g_{ij}^{(0)}(x)) < 0$. The above line element is assumed to be a solution of the *vacuum* Einstein field equations.

Upon performing the KK reduction with respect to the y^i -directions, the dimensionally reduced action corresponds to a non-linear $(D - p - 1)$ -dimensional σ -model with the target space $SL(p+1, R)/SO(p+1) \times SL(2, R)/SO(2) \times R$. These symmetries imply the existence of a Harrison transformation nontrivially acting on the spacetime variables and matter fields. Therefore one may generate nontrivial solutions of the equations (2.2) starting with known vacuum configurations. A detailed description of this procedure and the explicit form of the Harrison transformation at the σ -model level is given in ref. [7].

As a result, one finds a nonvacuum solution of the equations (2.2), with the line-element

$$ds^2 = \Omega^{-2\alpha(D-p-3)}g_{ij}^{(0)}(x)dy^i dy^j + \Omega^{2\alpha(p+1)}g_{\alpha\beta}^{(0)}(x)dx^\alpha dx^\beta, \quad (2.5)$$

the dilaton field and the only nonvanishing component of the $(p+1)$ -form being

$$\phi = \phi_0 - 2a\alpha(D-2)\log\Omega, \quad (2.6)$$

$$\mathcal{A}_{ty_1..y_p} = \sqrt{2(D-2)\alpha}e^{a\phi_0}|g_0|\Omega^{-1}U, \quad (2.7)$$

where

$$\alpha = [2a^2(D-2) + (p+1)(D-p-3)]^{-1}, \quad \text{and} \quad \Omega = (1-U^2)^{-1}(1-U^2|g_0|), \quad (2.8)$$

U being the parameter of the Harrison transformation, with $0 \leq U < 1$. Here ϕ_0 is a real constant corresponding to the asymptotic value of the scalar field; in what follows we shall set $\phi_0 = 0$, without any loss of generality.

In this way, one finds a family of solutions of the Einstein-dilaton-antisymmetric form equations, parameterized in terms of U . As we shall see later, the parameter U physically corresponds to the electrostatic potential difference between the horizon and infinity. One should also remark at this point that the new line element (2.5) preserves the symmetries of the seed metric (2.4).

2.3 The metric ansatz and asymptotics

The procedure described above works for any vacuum solution of the Einstein equation that can be written in the form (2.4), regardless of its asymptotic expression. However, a $(p+1)$ -form gauge potential naturally couples to the world volume of a p -brane. We recall that the black p -brane solutions are rich dynamical systems, exhibiting new behaviours that have no analogues in the black hole case [16].

In this work we consider solutions approaching at infinity the Minkowski spacetime \mathcal{M}^{D-p-1} times a Ricci flat space $T^p \times S^1$. The metric ansatz that we consider is

$$ds^2 = g_{tt}(r, z)dt^2 + g_{rr}(r, z)dr^2 + g_{zz}(r, z)dz^2 + g_{\omega\omega}(r, z)d\omega_{D-p-3}^2 + g_{yy}(r, z)\sum_{i=1}^p(dY^i)^2, \quad (2.9)$$

with r the radial coordinate in \mathcal{M}^{D-p-1} spacetime and $d\omega_d^2$ is the unit metric on S^d . Each Y^i -direction is compactified on a circle of length L_i (we denote $\mathcal{V}_p = \prod_{i=1}^p L_i$). z is also a periodic coordinate with period L that parametrizes S^1 , while t is the time coordinate. For any topology of the event horizon, the metric functions possess the following asymptotic behavior, with a power law decay¹

$$g_{tt} \simeq -1 + \frac{c_t}{r^{D-p-4}}, \quad g_{yy} \simeq 1 + \frac{c_y}{r^{D-p-4}}, \quad g_{zz} \simeq 1 + \frac{c_z}{r^{D-p-4}}, \quad g_{\omega\omega} \simeq r^2(1 + \mathcal{O}(1/r)), \quad (2.10)$$

c_t , c_y and c_z being real constants, which as discussed in the next section, encode the global charges of the solutions.

¹The situation is more complicated in $D-p=5$ dimensions for black objects with an explicit dependence on the z -direction. In these cases there are logarithmic terms in the asymptotic expansion of the metric functions. For $g_{\omega\omega}$, this occurs already at the next to leading order term, $g_{\omega\omega} \simeq r^2(1 + u \log r/r)$, with u a term that depends on c_t and c_z . However, the first two terms in the expression of g_{tt} , g_{zz} and g_{yy} are still given by (2.10). More generally one can have different constants c_y for each Y -direction; however this will not be the case for the solutions considered in this paper.

The generic asymptotic expression of the electric $(p + 1)$ -form is given by

$$\mathcal{A}_{t y_1 \dots y_p} = \Phi - \frac{q_e}{r^{D-p-4}} + \dots, \quad (2.11)$$

while the dilaton field decays as

$$\phi = -\frac{q_d}{r^{D-p-4}} + \dots, \quad (2.12)$$

with q_e, q_d real constants.

3. Conserved charges and the quasilocal formalism

3.1 The Mann-Marolf counterterm

In general, the conserved quantities of black P -brane spacetimes are encoded in the asymptotic expansions of the metric functions, as given for instance in (2.10). Their computation is a problem discussed by many authors, starting with [10] (see also [11, 12, 13]), mainly by making use of a reference background, whose obvious choice in the case of a general P -brane is $\mathcal{M}^{D-P} \times T^P$, where \mathcal{M}^{D-P} is the $(D - P)$ -dimensional Minkowski spacetime (to simplify the relations, we denote in this Section $P = p + 1$).

An alternative procedure motivated by the AdS/CFT conjecture and referred to as the ‘counterterm method’, which avoids the difficulties encountered in the background subtraction method, makes use of the quasilocal stress-tensor of Brown and York [17]. In this approach one supplements the action by including a suitable boundary counterterm:

$$I_{ct} = \frac{1}{8\pi G} \int_{\partial M} d^{D-1}x \sqrt{-h} \hat{K}. \quad (3.1)$$

As in the better known AdS case, the counterterms \hat{K} are functionals only of curvature invariants of the induced metric on the boundary and so they do not affect the equations of motion. Conserved quantities can be constructed using a boundary stress-tensor T_{ij} obtained via the algorithm of Brown and York, that is by taking the variation of the total action (including the counterterms), with respect to the boundary metric h_{ij} . Provided that the boundary geometry has an isometry generated by a Killing vector ξ^i , a conserved charge

$$\mathcal{Q}_\xi = \oint_\Sigma d^{D-2}S^i \xi^j T_{ij} \quad (3.2)$$

can be associated with a closed surface Σ . Physically, this means that a collection of observers on the boundary with the induced metric h_{ij} measure the same value of \mathcal{Q}_ξ . Although the counterterm method gives results that are equivalent to those obtained using the background subtraction method² [14, 9, 18], we employ it here because it appears to be a more general technique than background subtraction, and moreover, it is interesting to explore the range of problems to which it applies, in particular for configurations with a dependence on the compact extra-dimensions.

²When one can apply the background subtraction method.

Recently, a particularly interesting counterterm for asymptotically locally flat spacetimes has been put forward by Mann and Marolf [9] (see also [18, 19, 20, 21, 22]). This counterterm is taken to be the trace \hat{K} of a symmetric tensor \hat{K}_{ij} , which in general dimensions³ is defined implicitly in terms of the Ricci tensor \mathcal{R}_{ij} of the induced metric on the boundary via the relation:

$$\mathcal{R}_{ij} = \hat{K}_{ij}\hat{K} - \hat{K}_i^k\hat{K}_{kj}. \quad (3.3)$$

In solving this equation one picks a solution \hat{K}_{ij} that asymptotes to the extrinsic curvature of the boundary of the Minkowski space as the manifold boundary ∂M is pushed to infinity. One should note here that the boundary data $(\partial\mathcal{M}, h)$ is not uniquely defined by the bulk spacetime (\mathcal{M}, g) but it also depends on the limiting procedure used to define the Gibbons-Hawking term. As shown in [20], if one employs the so-called ‘cylindrical cut-off’, then the renormalized action leads to a boundary stress-tensor whose leading-order expression in $D > 4$ dimensions involves the electric part of the Weyl tensor:

$$T_{ij} = -\frac{1}{8\pi G}\frac{rE_{ij}}{D-3}, \quad (3.4)$$

where E_{ij} is the pull-back to the boundary $\partial\mathcal{M}$ of the contraction of the bulk Weyl tensor $C_{\mu\nu\rho\tau}$ with the induced metric $h_{\mu\nu}$. More precisely, introducing the unit normal vector n^μ to the boundary $\partial\mathcal{M}$ then the electric part of the bulk Weyl tensor is defined by:

$$E_{\mu\nu} = C_{\mu\rho\nu\tau}n^\rho n^\tau = -C_{\mu\rho\nu\tau}h^{\rho\tau}, \quad (3.5)$$

while E_{ij} is simply the pull-back to the boundary of the above tensor. In the ‘cylindrical cut-off’, which is more suitable for practical computations, the asymptotic form of the metric at spatial infinity takes the form:

$$ds^2 = -\left(1 + \mathcal{O}(r^{-(D-3)})\right)dt^2 + \left(1 + \mathcal{O}(r^{-(D-3)})\right)dr^2 + r^2\left(\omega_{IJ} + \mathcal{O}(r^{-(D-3)})\right)d\theta^I d\theta^J + \mathcal{O}(r^{-(D-4)})d\theta^I dt, \quad (3.6)$$

where ω_{IJ} and θ^I are the metric and coordinates on the unit $(D-2)$ -sphere.

Our aim here is to generalize these results and compute the action and conserved charges for the general P -branes considered in the previous Section. These spacetimes are not asymptotically flat and they belong to the class of the so-called ‘transverse asymptotically flat’ spacetimes. In general, one can compute the conserved mass, angular momenta and brane tensions for these spaces by using background subtraction procedures if one takes the reference background of the form $\mathcal{M}^{D-P} \times T^P$ [11, 12, 13]. Since the background spacetime is flat, one expects to be able to extend the construction of the Mann-Marolf counterterm to apply for this case as well and, following the steps in [20], we shall proceed to do so.

³In four dimensions its explicit form has been recently derived by Visser [21].

Generalizing the ansatz in (3.7) to apply for P -branes in D -dimensions, one takes the asymptotic form of the metric at spatial infinity of the following form:

$$ds^2 = - \left(1 + \mathcal{O}(r^{-(D-P-3)})\right) dt^2 + \left(1 + \mathcal{O}(r^{-(D-P-3)})\right) dr^2 + \sum_{a=1}^P \left(1 + \mathcal{O}(r^{-(D-P-3)})\right) (dy^a)^2 \\ + \sum_{a \neq b} \mathcal{O}(r^{-(D-P-4)}) dy^a dy^b + r^2 \left(\omega_{IJ} + \mathcal{O}(r^{-(D-P-3)})\right) d\theta^I d\theta^J + \mathcal{O}(r^{-(D-P-4)}) d\theta^I dt, \quad (3.7)$$

where ω_{IJ} and θ^I are the metric and coordinates on the unit $(D-P-2)$ -sphere. Here r plays the role of a ‘radial’ coordinate such that spacelike infinity is reached in the $r \rightarrow \infty$ limit and the symbols $\mathcal{O}(r^{-(D-P-3)})$ refer to terms that fall-off at least as fast as $r^{-(D-P-3)}$ in the $r \rightarrow \infty$ limit.

Note that the extrinsic curvature K_{ij} satisfies the Gauss-Codazzi relation:

$$\mathcal{R}_{ij} = R_{ikjl} h^{kl} + K_{ij} K - K_i^m K_{jm}, \quad (3.8)$$

where R_{ikjl} is the pull-back to the boundary $\partial\mathcal{M}$ of the bulk Riemann tensor. The leading terms of K_{ij} and \hat{K}_{ij} are given by the extrinsic curvature of the standard cylinder of radius r in the reference background $\mathcal{M}^{D-P} \times T^P$:

$$K_{ij} \sim \hat{K}_{ij} = r\omega_{ij} + \mathcal{O}(r^{-(D-P-4)}), \quad (3.9)$$

where ω_{ij} is the pull-back to $\partial\mathcal{M}$ of the round metric ω_{IJ} on the unit sphere S^{D-P-2} .

As shown in [9, 18], in order to compute the difference $\Delta_{ij} \equiv K_{ij} - \hat{K}_{ij}$ to first order one has to solve the linearized Gauss-Codazzi relation:

$$R_{ikjl} h^{kl} = -\Delta_{kl} h^{kl} \hat{K}_{ij} - \Delta_{ij} \hat{K} + \Delta_{il} \hat{K}_j^l + \Delta_{jl} \hat{K}_i^l. \quad (3.10)$$

Recall that h_{ij} is the induced metric on a hypersurface $r = \text{const.}$ with normal n^i such that $g_{ij} = h_{ij} + n_i n_j$. It is convenient at this stage to further define $\mu_{ij} = h_{ij} + u_i u_j - \sum_{a=1}^P y_i^a y_j^a$, where u^i is a future-directed timelike vector on $\partial\mathcal{M}$, associated with a foliation $\Sigma_t = \{t = \text{const.}\}$ of the spacetime \mathcal{M} , respectively $(y^a)^i$ is an outward pointing spacelike vector associated with a foliation $\Sigma_{y^a} = \{y^a = \text{const.}\}$ of \mathcal{M} . Then $\hat{K}_{ij} = \frac{\mu_{ij}}{r} + \mathcal{O}(r^{-(D-P-5)})$, while $\hat{K} = \frac{D-P-2}{r} + \mathcal{O}(r^{-(D-P-5)})$ and replacing these relations in (3.10) one obtains:

$$\frac{D-P-4}{r} \Delta_{ij} - \frac{\tilde{\Delta}_i u_j + \tilde{\Delta}_j u_i}{r} + \sum_{a=1}^P \frac{(\bar{\Delta}_i^a y_j^a + \bar{\Delta}_j^a y_i^a)}{r} \\ + \frac{\Delta}{r} (h_{ij} + u_i u_j - \sum_{a=1}^P y_i^a y_j^a) = -R_{ikjl} h^{kl}, \quad (3.11)$$

where we denote $\tilde{\Delta}_i = \Delta_{ij} u^j$, $\tilde{\Delta} = \Delta_{ij} u^i u^j$, respectively $\bar{\Delta}_i^a = \Delta_{ij} y^{aj}$ and $\bar{\Delta}^{ab} = \Delta_{ij} y^{ai} y^{bj}$. Quite generally, we shall consider matter sources for which the stress-energy tensor at infinity falls-off fast enough such that the Riemann and Weyl tensors agree to leading

order at spacelike infinity (this is the case of the solutions in this work). Therefore, one can replace in the right-hand side of the above equation the Riemann tensor by the Weyl tensor such that the right-hand side becomes equal to the electric part of the Weyl tensor E_{ij} .

Contracting now (3.11) with u^j one obtains:

$$\frac{D-P-3}{r}\tilde{\Delta}_i - \frac{\tilde{\Delta}u_i}{r} + \sum_{a=1}^P \frac{\tilde{\Delta}^a y_i^a}{r} = E_{ij}u^j \quad (3.12)$$

and further contractions with u^i , respectively y^{bi} leads to:

$$\tilde{\tilde{\Delta}} = \frac{r}{D-P-2}E_{ij}u^i u^j, \quad \tilde{\tilde{\Delta}}^a = \frac{r}{D-P-2}E_{ij}u^i y^{aj} \quad (3.13)$$

respectively. Similarly, contracting (3.11) successively with y^{ai} and y^{bj} leads to:

$$\bar{\tilde{\Delta}}^{ab} = \frac{r}{D-P-2}E_{ij}y^{ai}y^{bj}. \quad (3.14)$$

Finally, taking the trace of (3.11) with h^{ij} one obtains:

$$\Delta = \frac{\tilde{\tilde{\Delta}} - \sum_{a=1}^P \tilde{\tilde{\Delta}}^{aa}}{D-P-3}. \quad (3.15)$$

One is now ready to compute the boundary action (for vacuum spacetimes this is the total action):

$$\begin{aligned} I_{\partial\mathcal{M}} &= -\frac{1}{8\pi G} \int_{\partial M} d^{D-1}x \sqrt{-h}(K - \hat{K}) \\ &= -\frac{1}{8\pi G} \int_{\partial M} d^{D-1}x \sqrt{-h} \frac{r}{(D-P-2)(D-P-3)} E_{ij}(u^i u^j - \sum_{a=1}^P y^{ai} y^{aj}). \end{aligned} \quad (3.16)$$

Now, in order to compute the conserved charges we evaluate the boundary stress-tensor using the formula (see eq. (4.2) of [9]):

$$T_{ij} = -\frac{1}{8\pi G}(\Delta_{ij} - h_{ij}\Delta), \quad (3.17)$$

from which the conserved charges associated with a boundary Killing vector ξ are found by direct integration as in (3.2). The conserved mass is generally defined using an asymptotic time translation Killing vector $\xi = \partial/\partial t$, while the angular momenta are defined using angular Killing vectors $\xi = \partial/\partial\varphi$. If Σ_t is a spatial slice and $\partial\Sigma_t^\infty$ is its intersection with the boundary $\partial\mathcal{M}$ at spatial infinity, then one integrates (3.2) on $\partial\Sigma_t^\infty$ over the brane directions y^a and the S^{D-P-2} sphere at infinity to compute the mass and angular momenta. Therefore, to compute the conserved mass and angular momenta one basically has to evaluate the quantity:

$$\begin{aligned} 8\pi G T_{ij}u^j &= -(\tilde{\Delta}_i - \Delta u_i) \\ &= -\frac{r}{(D-P-2)(D-P-3)} [(D-P-2)E_{ij}u^j + \sum_{a=1}^P E_{lm}y^{al}(y^{am}u_i - u^m y_i^a)], \end{aligned} \quad (3.18)$$

where we made use of the formulae (3.12) and (3.15). As an example, the conserved mass will be given by:

$$\begin{aligned} M &= \frac{1}{8\pi G} \int d^P y d^{D-P-2} \theta \sqrt{\mu} T_{ij} \xi^i u^j \\ &= -\frac{\mathcal{V}_P}{8\pi G} \int d^{D-P-2} \theta \sqrt{\mu} \frac{r}{(D-P-2)(D-P-3)} [(D-P-2) E_{ij} \xi^i u^j - \sum_{a=1}^P E_{ij} y^{ai} y^{aj}], \end{aligned} \quad (3.19)$$

where $\xi = \partial/\partial t$ and $\mathcal{V}_P = \int d^P y$ is the volume along the brane directions. Similarly, the angular momentum will be computed using:

$$\begin{aligned} J &= \frac{1}{8\pi G} \int d^P y d^{D-P-2} \theta \sqrt{\mu} T_{ij} \xi^i u^j \\ &= -\frac{\mathcal{V}_P}{8\pi G} \int d^{D-P-2} \theta \sqrt{\mu} \frac{r}{(D-P-3)} E_{ij} \xi^i u^j, \end{aligned} \quad (3.20)$$

where now $\xi = \partial/\partial\phi$.

The gravitational tension tensor can be defined similarly using the asymptotic spatial translation Killing vectors $\partial/\partial y^a$ and the integral is now taken over all the spatial directions, excluding however the brane direction [23]. To compute the tension tensor density one has to evaluate the quantity:

$$\begin{aligned} 8\pi G T_{ij} y^{aj} &= -(\bar{\Delta}_i^a - \Delta y_i^a) \\ &= -\frac{r}{(D-P-2)(D-P-3)} [(D-P-2) E_{ij} y^{aj} + E_{lm} y^{al} (u^m u_i - \sum_{b=1}^P y^{bm} y_i^b) \\ &\quad - E_{lm} (u^l u^m - \sum_{b=1}^P y^{bl} y^{bm}) y_i^a]. \end{aligned} \quad (3.21)$$

In consequence, the tension tensor density (defined as in [23]) is given by:

$$\begin{aligned} \mathcal{T}^{ab} &= -\frac{1}{8\pi G} \int d^{D-P-2} \theta \sqrt{\mu} T_{ij} y^{ai} y^{bj} \\ &= \frac{1}{8\pi G} \int d^{D-P-2} \theta \sqrt{\mu} \frac{r}{(D-P-2)(D-P-3)} [(D-P-3) E_{ij} y^{ai} y^{bj} \\ &\quad - \delta^{ab} E_{ij} (u^i u^j - \sum_{c=1}^P y^{ci} y^{cj})]. \end{aligned} \quad (3.22)$$

Note that \mathcal{T}^{ab} is a symmetric tensor.

Suppose now that, regardless of the topology of the event horizon (if present), some of the metric functions appearing in (3.7) have the following asymptotic form⁴:

$$g_{tt} \simeq -1 + \frac{c_t}{r^{D-P-3}}, \quad g_{y^a y^a} \simeq 1 + \frac{c_a}{r^{D-P-3}}, \quad g_{y^a y^b} \simeq \frac{d_{ab}}{r^{D-P-3}}, \quad (3.23)$$

⁴Note that the asymptotics (3.23) are more general than (2.10), since extra-diagonal components in the y^a -directions are allowed. This would cover also the case of KK solutions with an arbitrary torus compactifications, so far considered in perturbation theory only [24].

where c_t, c_a and d_{ab} are real constants, with $a = 1..P$.

Then it can be checked that to leading order we have the following relations:

$$\begin{aligned} E_t^t &= \frac{(D-P-2)(D-P-3)}{2} \frac{c_t}{r^{D-P-1}}, & E_y^y{}^a &= -\frac{(D-P-2)(D-P-3)}{2} \frac{c_a}{r^{D-P-1}}, \\ E_b^a &= -\frac{(D-P-2)(D-P-3)}{2} \frac{d_{ab}}{r^{D-P-1}}, \end{aligned} \quad (3.24)$$

for any $a, b = 1..P$. One can then easily compute the conserved mass and the diagonal components of the tension tensor which are given by the relations (3.29) below. Note that in the general case, the tension tensor has also the non-diagonal components:

$$\mathcal{T}^{ab} = -\frac{\Omega_{D-P-3}}{16\pi G}(D-P-4)d_{ab}. \quad (3.25)$$

Recall now that we are interested in discussing the thermodynamic properties of black objects whose metrics are asymptotic to the form given in (2.9), which is appropriate when considering nonuniform extended objects with $(p+1)$ -compact spatial directions. In order to accommodate this demand one simply has to replace in the above formulae P with $p+1$ and single out one of the coordinates, say the $(p+1)$ -coordinate and denote it by z , while the p remaining compact directions will be denoted by Y^i . Further note that for the ansatz considered in (2.9) one has $c_a = c_y$, $d_{ab} = 0$ for $a, b = 1..p$ and $c_{p+1} = c_z$, while $\mathcal{V}_P = \mathcal{V}_p L$, where L is the length at infinity along the z -direction.

3.2 Another counterterm choice

For comparison and also as a check of the correctness of the results obtained in the previous section, we shall complement these results with those obtained by using the following boundary counterterm:

$$I_{ct} = \frac{c_{(D,p)}}{8\pi G} \int_{\partial\mathcal{M}} d^{D-1}x \sqrt{-h} \sqrt{\mathcal{R}}, \quad (3.26)$$

where $c_{(D,p)} = \sqrt{(D-p-3)/(D-p-4)}$, and \mathcal{R} is the Ricci scalar of the boundary geometry.

Varying the total action with respect to the boundary metric h_{ij} , we compute the boundary stress-tensor:

$$T_{ij} = \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h^{ij}} = -\frac{1}{8\pi G} \left(K_{ij} - h_{ij} K - \Psi(\mathcal{R}_{ij} - \mathcal{R}h_{ij}) - h_{ij} \square \Psi + \Psi_{;ij} \right), \quad (3.27)$$

where K_{ij} is the extrinsic curvature of the boundary and we denote $\Psi = c_{(D,p)} \sqrt{1/\mathcal{R}}$.

One should remark here that the counterterm choice is not unique, other choices being possible as well depending on the boundary topology (see [14, 25, 26, 27, 20] for a related discussion). Our choice of using (3.26) was motivated by the fact that the general expression for the boundary stress-tensor is very simple⁵.

⁵For example, we have verified that for $D = 5, 6, 7$, the same expression for the mass and tension are found by using a counterterm on the form (3.26), with $\sqrt{\mathcal{R}}$ replaced by $\mathcal{R}^{3/2}/\sqrt{\mathcal{R}^2 - \mathcal{R}_{ij}^{ij}}$. However, in that case, the expression of the boundary stress-tensor is much more complicated than (3.27), see e.g. ref. [27].

Using the counterterm (3.26) for geometries whose asymptotic form is given by (2.9), to leading order term the relevant components of the boundary stress-tensor are given by:

$$\begin{aligned} T_t^t &= -\frac{1}{16\pi G} \frac{(D-p-3)c_t - c_z - pc_y}{r^{D-p-3}}, \\ T_{Y_i}^{Y_i} &= -\frac{1}{16\pi G} \frac{c_t - c_z - (D-4)c_y}{r^{D-p-3}}, \\ T_z^z &= -\frac{1}{16\pi G} \frac{c_t - (D-p-3)c_z - pc_y}{r^{D-p-3}}. \end{aligned} \quad (3.28)$$

Note that we already made use of the relations $c_a = c_y$, for $a = 1..p$ and $c_{p+1} = c_z$. The mass and the gravitational tensions are the conserved charges (3.2) associated to $\partial/\partial t$, $\partial/\partial y^i$ and $\partial/\partial z$, respectively (note that $\partial/\partial z$ is an asymptotic Killing symmetry of the metric). For a straight torus geometry along the brane directions it becomes apparent that the tension-tensor \mathcal{T}^{ab} is diagonal. Keeping this in mind, when defining the gravitational tension \mathcal{T}^{aa} along one direction y^a it is also more convenient at this point to extend the integration along all the spatial directions excluding only the direction y^a . For a given direction y^a this amounts to multiplying \mathcal{T}^{aa} with the factor \mathcal{V}_P/L_a .

3.3 The global charges

For both the counterterm choices discussed above, one finds the following expressions for the gravitational charges:

$$\begin{aligned} M &= \frac{V_p L \Omega_{D-p-3}}{16\pi G} ((D-p-3)c_t - c_z - pc_y), \\ \mathcal{T}_z &= \frac{V_p \Omega_{D-p-3}}{16\pi G} (c_t - (D-p-3)c_z - pc_y), \\ \mathcal{T}_{Y_i} &= \frac{L}{L_i} \frac{V_p \Omega_{D-p-3}}{16\pi G} (c_t - c_z - (D-4)c_y), \end{aligned} \quad (3.29)$$

where Ω_{D-p-3} is the area of the unit $(D-p-3)$ -dimensional sphere⁶.

Usually in the literature one also defines a relative tension

$$n_I = \frac{\mathcal{T}_I L_I}{M}, \quad (3.30)$$

(with $I = z, i$) which measures how large the tension along the I -direction (or binding energy) is relative to the mass.

The solutions we construct will also have an electric charge which is computed according to

$$Q_e = \frac{\mathcal{V}_p L \Omega_{D-p-3}}{16\pi G} \int e^{a\phi} \star F = \frac{V_p L \Omega_{D-p-3}}{16\pi G} (D-p-4) q_e, \quad (3.31)$$

(with q_e defined in (2.11)), and a dilaton charge Q_d which is read from the asymptotic expansion (2.12) of the scalar field

$$Q_d = \frac{\mathcal{V}_p L \Omega_{D-p-3}}{16\pi G} (D-p-4) q_d. \quad (3.32)$$

⁶Note that these results are consistent with those previously derived (by other means) in the literature [10]-[13].

As we shall discuss in the next Section, Q_d is not an independent charge for the solutions generated by using the Harrison transformation and it can be expressed in terms of mass, z -tension and electric charge. The same holds for the gravitational tensions along the Y^i -directions.

4. KK static, vacuum black objects in $d = D - p$ dimensions

The charged black objects discussed in the next sections are constructed using as seed solutions the known vacuum static BHs and black strings in $d = D - p$ dimensions (*i.e.* with only one compact direction). In absence of matter fields (or of a cosmological constant), these configurations are trivially uplifted to D -dimensions by adding p -flat directions Y^i . Therefore it is of interest here to briefly review their basic properties (see [28] for a more detailed overview of the KK solutions).

Already in this case, the vacuum Einstein equations admit a variety of interesting solutions, approaching asymptotically the $\mathcal{M}^{d-1} \times S^1$ background. Of interest here are the black strings and the black holes, which are distinguished according to their horizon topology. Both types of solutions are usually described within the following ansatz with three unknown functions⁷ A, B, C :

$$ds_d^2 = -e^{2A(r,z)} f(r) dt^2 + e^{2B(r,z)} \left(\frac{dr^2}{f(r)} + dz^2 \right) + e^{2C(r,z)} R^2(r,z) d\omega_{d-3}^2, \quad (4.1)$$

where A, B, C vanish at large coordinate r . $f(r)$ and $R(r,z)$ are suitable background functions. Usually one takes $f(r) = 1 - (r_0/r)^{d-4}$, $R(r,z) = r$ for black strings, in which case $r_0 > 0$ corresponds to the horizon radius. The usual choice for black holes is $f(r) = 1$, while $R(r,z)$ has a complicated form (with $R(r,z) \rightarrow r$ in the asymptotic region), see Appendix A.

Unfortunately the only solutions known in closed form (except for the bubbles, which we do not consider in this work) have no z -dependence and describe uniform black strings (UBSs) with $A = B = C = 0$ (*i.e.* the direct product of the Schwarzschild-Tangherlini solution in $(d-1)$ -dimensions with a spacelike circle). The existence of both caged BHs and NUBSs has been asserted so far mainly based on numerical results (note, however, that analytic approximations of the small BH solutions have been presented in [30]). Such solutions have been found at the nonperturbative level by solving a set of three partial differential equations for the metric functions A, B, C , subject to a suitable set of boundary conditions. The technical aspects involved in the numerical construction of these solutions are presented in Appendix A.

The horizon topology of the black string solutions is $S^{d-3} \times S^1$. As shown by Gregory and Laflamme [31], a uniform solution is unstable below a critical value of the mass. As a result, there also exist nonuniform strings, which emerge smoothly from the UBS branch at a critical point, where its stability changes [31]. Keeping fixed the horizon coordinate

⁷An ansatz involving only two undetermined functions has been proposed in [29], for a special choice of coordinates. However, it remains an interesting open problem whether one can use that ansatz for numerical calculations.

r_0 and the asymptotic length L of the compact direction, these solutions depend on a single parameter, specified via the boundary conditions and physically corresponding to the Hawking temperature of the solutions [32]. Varying this parameter, the nonuniform strings become increasingly deformed (see Appendix A for details). A measure of the deformation of the geometry of these solutions is given by the nonuniformity parameter λ [32]

$$\lambda = \frac{1}{2} \left(\frac{\mathcal{R}_{\max}}{\mathcal{R}_{\min}} - 1 \right), \quad (4.2)$$

where \mathcal{R}_{\max} and \mathcal{R}_{\min} represent the maximum radius of a $(d-3)$ -sphere on the horizon and the minimum radius, being the radius of the ‘waist’. Concerning the thermodynamical stability of the NUBS solutions, it turns out that, at least for $d=5, 6$, the entropy decreases with temperature, *i.e.* they are thermally unstable.

Apart from the black strings, KK theory also admits as solutions a branch of caged black holes with an event horizon of topology S^{d-2} . For the metric ansatz (4.1), the event horizon is still located at a constant value of the radial coordinate, $r = r_0$, with $-\infty < r < 0$. (Note that the functions A, B, C contain in this case suitable ‘background’ parts [33] that enforce the required behaviour on the boundaries and the topology difference between the event horizon and infinity, see Appendix A.)

For small values of the mass, these solutions look like the Schwarzschild-Tangerlini solution in d dimensions. Once one increases their horizon radius for a fixed value of the asymptotic length L of the extra-direction, they deform as they start to ‘feel’ the compact S^1 direction and their mass increases. Analogous to the NUBS case, the geometrical properties of the black holes can be described in terms of two parameters denoted here by R_{eq} , the equatorial radius of the horizon, and L_{axis} , the proper length along the exposed symmetry axis. Again, the entropy of the caged BHs is a decreasing function of temperature, *i.e.* the considered BHs are thermally unstable⁸.

Although a number of details remain to be clarified, the numerical results suggest that the black hole and the NUBS branches merge at a topology changing transition [6], [35]. This scenario is supported by all numerical data available. For NUBS, the normalized thermodynamic quantities change between one (the uniform solution) and some critical values. The normalized mass, relative tension and entropy of the caged black holes vary between zero and some critical values that seem to correspond to those where the NUBS branch stops to exist. At the same time, the Hawking temperature of the BHs varies between infinity (for vacuum state with $M = T = 0$) and a minimal value. The set of black hole and the set of NUBS solutions then appear to merge at critical values of these quantities. In a (n, M) or (n, T) diagram, this feature can be seen looking at the vacuum data ($U = 0$) in Figure 5. (For convenience, the thermodynamic quantities there are normalized by their value for the critical UBS solution.) This scenario also implies that the ‘waist’ of the NUBSs would shrink to nothing, locally forming a cone geometry. At that point, both L_{axis} and R_{\min} go to zero, while R_{eq} tends to R_{\max} .

⁸Note however, that the recent results in [34] indicate the existence of a more complicated picture, with the possible existence of caged BHs possessing a positive specific heat.

For both black holes and black strings, the following Smarr relation holds

$$\frac{(d-3)M - L\mathcal{T}}{d-2} = T_H S, \quad (4.3)$$

where T_H is the Hawking temperature and S is the entropy of the solutions (which is one quarter of the horizon area). The first law of thermodynamics, which for the KK black objects takes the form:

$$dM = T_H dS + \mathcal{T} dL \quad (4.4)$$

is also satisfied.

5. Charged black objects in KK theory

The vacuum configurations discussed above can be uplifted to D -dimensions and used to generate solutions of the Einstein-dilaton-antisymmetric form theory by applying a Harrison transformation, as described in Section 2. The asymptotic structure of the metric and the topology of the event horizon remain unchanged in the process.

5.1 The solutions and the geometry

In our approach, the two metric parts which enter (2.4) are

$$\begin{aligned} g_{ij}^{(0)}(x)dy^i dy^j &= -e^{2A_0(r,z)} f(r) dt^2 + \sum_{i=1}^p (dY^i)^2, \\ g_{\alpha\beta}^{(0)}(x)dx^\mu dx^\nu &= e^{2B_0(r,z)} \left(\frac{dr^2}{f(r)} + dz^2 \right) + e^{2C_0(r,z)} R^2(r,z) d\omega_{D-p-3}^2, \end{aligned} \quad (5.1)$$

where $A_0(r,z)$, $B_0(r,z)$ and $C_0(r,z)$ are the functions of a vacuum seed solution.

The simplest charged solution one can construct by employing the Harrison transformation described in Section 2, uses a vacuum uniform black string solution as seed configuration. The resulting solution corresponds to a $(p+1)$ -brane solution with the line element:

$$\begin{aligned} ds^2 = & - \left(1 + \left(\frac{r_0}{r} \right)^{D-p-4} \frac{U^2}{1-U^2} \right)^{-2\alpha(D-p-3)} \left[- (1 - \left(\frac{r_0}{r} \right)^{D-p-4}) dt^2 + \sum_{i=1}^p (dY^i)^2 \right] \\ & + \left(1 + \left(\frac{r_0}{r} \right)^{D-p-4} \frac{U^2}{1-U^2} \right)^{2\alpha(p+1)} \left(\frac{dr^2}{1 - \left(\frac{r_0}{r} \right)^{D-p-4}} + dz^2 + r^2 d\omega_{D-p-3}^2 \right), \end{aligned} \quad (5.2)$$

and matter fields

$$\begin{aligned} \mathcal{A}_{ty_1 \dots y_p} &= \sqrt{2(D-2)\alpha} \frac{\left(1 - \left(\frac{r_0}{r} \right)^{D-p-4} \right)}{1 + \left(\frac{r_0}{r} \right)^{D-p-4} \frac{U^2}{1-U^2}} U, \\ \phi &= -2a(D-2)\alpha \log \left(1 + \left(\frac{r_0}{r} \right)^{D-p-4} \frac{U^2}{1-U^2} \right), \end{aligned}$$

with α given by (2.8) and $0 \leq U < 1$. For any value of a, U , the surface $r = r_0$ is still an event horizon, while $r = 0$ is a curvature singularity. As we have previously mentioned, the geometric structure outside the event horizon and also the asymptotic structure are

preserved by the solution generating procedure. The inclusion of the matter fields affects the geometry in both z - and Y^i -directions, however in different ways. One can see that the proper length of the z circle increases for any finite r , while the opposite holds for a Y^i -direction. Moreover, for a given value of r , the area Ω_{D-p-3} of the $D-p-3$ -sphere increases with U .

The extremal limit of (5.2) is found by taking $U \rightarrow 1$ together with a rescaling of r_0 such that $r_0^{D-p-4} = (1 - U^2)c^{D-p-4}$ and it reads:

$$ds^2 = -\left(1 + \left(\frac{c}{r}\right)^{D-p-4}\right)^{-2\alpha(D-p-3)} \left(-dt^2 + \sum_{i=1}^p (dY^i)^2\right) \\ + \left(1 + \left(\frac{c}{r}\right)^{D-p-4}\right)^{2\alpha(p+1)} (dr^2 + dz^2 + r^2 d\omega_{D-p-3}^2), \quad (5.3)$$

$$\mathcal{A}_{ty_1\dots y_p} = \sqrt{2(D-2)\alpha} \left(1 + \left(\frac{c}{r}\right)^{D-p-4}\right)^{-1}, \quad \phi = -2a(D-2)\alpha \log(1 + \left(\frac{c}{r}\right)^{D-p-4}), \quad (5.4)$$

c being a positive constant.

The basic features of the nonextremal uniform configurations are shared by the charged solutions generated starting from NUBS and caged black holes. In both cases the geometry is described by the metric form:

$$ds^2 = -e^{2A(r,z)} f(r) dt^2 + e^{2B(r,z)} \left(\frac{dr^2}{f(r)} + dz^2\right) \\ + e^{2C(r,z)} R^2(r,z) d\omega_{D-3}^2 + e^{2C_1(r,z)} \sum_{i=1}^p (dY^i)^2, \quad (5.5)$$

where

$$A(r,z) = A_0(r,z) - \alpha(D-p-3) \log \Omega(r,z), \quad B(r,z) = B_0(r,z) + \alpha(p+1) \log \Omega(r,z), \\ C(r,z) = C_0(r,z) + \alpha(p+1) \log \Omega(r,z), \quad C_1(r,z) = -\alpha(D-p-3) \log \Omega(r,z), \quad (5.6)$$

and

$$\Omega(r,z) = \frac{1}{1-U^2} (1 - U^2 e^{2A_0(r,z)} f(r)). \quad (5.7)$$

The only nonvanishing component of the $(p+1)$ -form and the dilaton are given by

$$\mathcal{A}_{ty_1\dots y_p} = \sqrt{2(D-2)\alpha} e^{2A_0(r,z)} f(r) \frac{\Omega(r,z)}{U}, \quad \phi = -2a(D-2)\alpha \log \Omega(r,z). \quad (5.8)$$

Again, the position of the horizon is not affected by the Harrison transformation. Once one knows the expressions of the functions $A_0(r,z)$, $B_0(r,z)$, $C_0(r,z)$ for the vacuum solution, one can derive all the geometric properties of the corresponding charged configurations. However, in absence of an explicit analytical solution, it is unclear how to derive the extremal version of the black objects with an explicit z -dependence starting from the non-extremal solution (5.5), (5.8). Therefore, for the rest of this work we shall consider nonextremal solutions only.

5.2 The charges of the solutions

A straightforward computation based on the generic asymptotic form of the metric functions (2.10) leads to the simple relations:

$$\begin{aligned} c_t &= c_t^{(0)} \left(1 + 2\alpha(D-p-3)\frac{U^2}{1-U^2}\right), & c_z &= c_z^{(0)} + 2c_t^{(0)}\alpha(p+1)\frac{U^2}{1-U^2}, \\ c_y &= -2\alpha(D-p-3)c_t^{(0)}\frac{U^2}{1-U^2}, \end{aligned} \quad (5.9)$$

$c_t^{(0)}$, $c_z^{(0)}$ being the constants in the asymptotic expansion of the seed solution with $U = 0$. The expressions for the electric charge Q_e and the dilaton charge Q_d , as derived from (2.10),(3.31),(3.32), are:

$$Q_e = \frac{LV_p\Omega_{D-p-3}}{16\pi G}(D-p-4)\sqrt{2(D-2)\alpha}\frac{U}{1-U^2}c_t^{(0)}, \quad (5.10)$$

$$Q_d = \frac{LV_p\Omega_{D-p-3}}{16\pi G}(D-p-4)\frac{U^2}{1-U^2}c_t^{(0)}. \quad (5.11)$$

There is also a one-to-one correspondence between the solution generating parameter U and the electrostatic potential difference Φ between the event horizon and infinity

$$\Phi = \sqrt{2(D-2)\alpha}U. \quad (5.12)$$

As argued below, in a thermodynamical description of the system, Φ corresponds to the chemical potential.

One can see that the mass and tensions of a charged solution can be expressed as linear combinations of the corresponding quantities of the vacuum configuration:

$$\begin{bmatrix} M \\ \mathcal{T}_{y^i} \\ \mathcal{T}_z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} M_0 \\ \mathcal{T}_{y^i}^0 \\ \mathcal{T}_z^0 \end{bmatrix}, \quad (5.13)$$

where

$$\begin{aligned} a_{11} &= 1 + \frac{2(D-2)(D-p-3)\alpha}{(D-p-2)}\frac{U^2}{1-U^2}, & a_{13} &= -\frac{2(D-2)\alpha L}{(D-p-3)}\frac{U^2}{1-U^2}, \\ a_{21} &= \frac{2(D-2)(D-p-3)\alpha}{(D-p-2)L_i}\frac{U^2}{1-U^2}, & a_{23} &= -\frac{2(D-2)(D-p-3)\alpha L_i}{(D-p-2)L}\frac{U^2}{1-U^2}, \end{aligned}$$

while $a_{12} = a_{31} = a_{32} = 0$, $a_{22} = a_{33} = 1$. (Note that the tension along the z -direction does not change when adding an electric charge to the system).

The electric charge can also be expressed as a function of mass and z -tension of the vacuum solution:

$$Q_e = \frac{\sqrt{2(D-2)\alpha}}{(D-p-2)} \left((D-p-3)M_0 - LT_z^0\right) \frac{U}{1-U^2}, \quad (5.14)$$

with a similar relation holding for the dilaton charge.

The initial vacuum configuration has two global charges, M_0 and \mathcal{T}_z^0 . The Harrison transformation introduces via the parameter U a new charge. This implies that Q_d and \mathcal{T}_i are not new global independent charges and therefore they do not enter the thermodynamical description of the system since they cannot be varied at will. Indeed, one finds the following quadratic relation

$$\frac{Q_e^2}{\kappa_1(M(D-p-3)-L\mathcal{T}_z)+Q_d} = \left(1 - \frac{2(D-2)(D-p-3)\alpha}{D-p-2}\right) Q_d, \quad (5.15)$$

with

$$\kappa_1 = \frac{\sqrt{2(D-2)\alpha}}{(D-p-4)(D-p-2-2(D-2)(D-p-3)\alpha)}, \quad (5.16)$$

which determines the dilaton charge in terms of the mass, tension along the z -direction and the electric charge. The tensions along the Y^i -directions can also be written as

$$L_i \mathcal{T}_i = \frac{1}{D-p-3} \left(\frac{2(D-2)(D-p-3)U^2\alpha + 1 - U^2}{2(D-2)U^2\alpha} \Phi Q_e + L\mathcal{T}_z \right). \quad (5.17)$$

5.3 The thermodynamics

The Hawking temperature T_H of a black object is computed either by requiring the absence of conical singularities on the Euclidean section (which is found by analytical continuation $t \rightarrow i\tau$) or from the surface gravity of the solutions in the Lorentzian section. As usual with black objects in Einstein gravity, the KK solutions considered in this work have an entropy which is one quarter of the event horizon area.

One can easily prove that the relation between Hawking temperature T_H and the entropy S of the charged solutions and the corresponding quantities $T_{H(0)}$ and S_0 of the vacuum seed solution is

$$T_H = T_{H(0)}(1-U^2)^{\alpha(D-2)}, \quad S = S_0(1-U^2)^{-\alpha(D-2)}, \quad (5.18)$$

such that the product between the entropy and temperature remains invariant under the Harrison transformation. Thus adding an electric charge to a system with a fixed event horizon radius would decrease its temperature and increase the entropy, which agrees with the experience based on Reissner–Nordström solution.

Considering the thermodynamics of these solutions, they should satisfy the first law of thermodynamics

$$dM = T_H dS + \Phi dQ_e + \mathcal{T}_z dL. \quad (5.19)$$

One may regard the parameters S , Q_e and L as a complete set of extensive parameters for the mass $M(S, Q_e, L)$ and define the intensive parameters conjugate to them. These quantities are the temperature, the chemical potential and the z -tension.

For any type of charged black object, the following Smarr relation holds

$$\frac{(D-p-3)M - \mathcal{T}_z L}{D-p-2} = T_H S + \frac{D-p-3}{D-p-2} \Phi Q_e, \quad (5.20)$$

which results from (4.3) together with (5.13), (5.14) and (5.18).

In the canonical ensemble, we study solutions holding the temperature T_H , the electric charge Q_e and the length L of the extra-dimension fixed. The associated thermodynamic potential is the Helmholtz free energy

$$F[T_H, Q_e, L] = M - T_H S, \quad dF = -SdT_H + \Phi dQ_e + T_z dL. \quad (5.21)$$

The situation of KK black objects in a grand canonical ensemble is also of interest, in which case we keep the temperature, the chemical potential and the tension fixed. In this case the thermodynamics is obtained from the Gibbs potential

$$G[T_H, \Phi, T_z] = E - T_H S - \Phi Q_e - T_z L. \quad (5.22)$$

By using the relations (5.13), (5.14), one finds that both F and G can also be expressed in terms of the global charges of the vacuum seed solutions and the generating parameter U .

It is well-known that different thermodynamic ensembles are not exactly equivalent and may not lead to the same conclusions regarding the thermodynamic stability as they correspond to different physical situations (see e.g. [36], [37]). Mathematically, thermodynamic stability is equated with the subadditivity of the entropy function. In the canonical ensemble, the subadditivity of the entropy is exactly equivalent to the positivity of the specific heat at constant (Q_e, L)

$$C_{Q,L} \equiv \left(\frac{\partial M}{\partial T_H} \right)_{Q,L} = T_H \left(\frac{\partial S}{\partial T_H} \right)_{Q,L} > 0. \quad (5.23)$$

Remarkably, one can prove that the specific heat of a charged configuration can be expressed in terms of the specific heat and entropy of the vacuum seed solution⁹:

$$C_{Q,L} = \frac{1}{(1-U^2)^{2\alpha(D-2)}} \frac{C_{Q,L}^0 (1+U^2 - 2\alpha(D-2)U^2) - 2\alpha(D-2)U^2 S_0}{U^2 + 1 + 2\alpha(D-2)U^2 (1 + \frac{C_{Q,L}^0}{S_0})}. \quad (5.24)$$

The condition for thermodynamic stability in the grand canonical ensemble is [37], [38]:

$$C_{Q,L} > 0, \quad \epsilon_T \equiv \left(\frac{\partial Q}{\partial \Phi} \right)_{T_H} > 0, \quad (5.25)$$

where ϵ_T is the isothermal electric permittivity, which measures the stability of solutions to electrical fluctuations. This quantity can also be expressed in terms of the quantities of the vacuum seed solution¹⁰

$$\epsilon_T = \frac{T_{H(0)} S_0}{(1-U^2)^2} \left(U^2 + 1 + 2\alpha(D-2)U^2 (1 + \frac{C_{Q,L}^0}{S_0}) \right). \quad (5.26)$$

⁹In evaluating $C_{Q,L}$ one uses (5.18) together with the expression (5.10) for the electric charge. The Smarr relation (4.3) implies also $c_t^{(0)} \sim T_{H(0)} S_0$. Thus, one finds $T_{H(0)} S_0 d(U/(1-U^2)) + U/(1-U^2) d(T_{H(0)} S_0) = 0$ and substitutes back in (5.23).

¹⁰Here one uses the relation $0 = dT_{H(0)} - 2\alpha(D-2)U/(1-U^2)T_{H(0)}dU$, which follows from (5.18).

We have seen in the previous Section that the specific heat of the vacuum solutions $C_{Q,L}^0$ is a strictly negative quantity for both black holes and black strings. However, one can see from (5.24), (5.26) that $C_{Q,L}$ and ϵ_T may become positive for some range of U , a, p .

Here we would like to point out that for any black object constructed by using the $d = 5, 6$ vacuum solutions discussed in Section 4, the condition (5.25) for thermodynamic stability in the grand canonical ensemble cannot be satisfied. To prove that, we express $C_{Q,L}$ and ϵ_T as

$$C_{Q,L} = \frac{1}{(1-U^2)^{2\alpha(D-2)}} S_0 \left(\frac{x+1}{1+\bar{\alpha}(x+1)} - 1 \right), \quad \epsilon_T = \frac{T_0 S_0}{(1-U^2)^2} (1+U^2)(1+\bar{\alpha}(x+1)), \quad (5.27)$$

where we denote $x = C_{Q,L}^0/S_0$, $\bar{\alpha} = \alpha(D-2)2U^2/(1+U^2)$. Moreover, we have verified¹¹ that $C_{Q,L}^0 + S_0 < 0$ for all vacuum seed solutions with $d = 5, 6$ (*e.g.* this quantity is $C_{Q,L}^0/S_0 + 1 = -(D-p-4)$ for UBSs). Therefore $x+1 < 0$ and, as a result, ϵ_T and $C_{Q,L}$ cannot be both positive at the same time. (This is a general result, valid for any values of a, p, U .) Thus we conclude that all charged KK solutions generated from $d = 5, 6$ vacuum solutions are unstable in a grand canonical ensemble.

Of course, this does not exclude the existence of stable solutions in a canonical ensemble. In the next subsection we shall find such configurations for both black holes and black strings (*i.e.* with $p = 0$). A similar analysis for a generic p does not seem to be possible, except for the case of uniform brane solutions (thus without an explicit dependence on z), which we shall consider next. The quantities which enter the thermodynamics can be derived¹² easily in this case from (5.2)

$$\begin{aligned} M &= \frac{\Omega_{D-p-4} V_p L}{16\pi G} ((D-p-3)(1-U^2) + 2(D-2)(D-p-4)U^2\alpha) \frac{r_0^{D-p-4}}{1-U^2}, \\ T_z &= \frac{\Omega_{D-p-4} V_p L}{16\pi G} r_0^{D-p-4}, \quad Q_e = \frac{\Omega_{D-p-4} V_p L}{16\pi G} \sqrt{2(D-2)}(D-p-4) \frac{U}{1-U^2} r_0^{D-p-4}, \end{aligned} \quad (5.28)$$

where r_0 is the event horizon radius. One can see that the charge to mass ratio of these solutions has a nontrivial dependence on a, p, U :

$$\frac{Q_e}{M} = \frac{\sqrt{2(D-2)\alpha}(D-p-4)U}{(D-p-3)(1-U^2) + 2(D-2)(D-p-4)U^2\alpha}. \quad (5.29)$$

The specific heat and isothermal electric permittivity of the uniform solutions are given by

$$C_{Q,L} = \frac{S_0}{(1-U^2)^{2\alpha(D-2)}} \frac{2\alpha(D-2)(D-p-4)U^2 - (1+U^2)(D-p-3)}{1+U^2 - 2\alpha(D-2)(D-p-4)U^2}, \quad (5.30)$$

$$\epsilon_T = \frac{T_0 S_0}{(1-U^2)^2} (U^2 + 1 - 2\alpha(D-2)(D-p-4)U^2). \quad (5.31)$$

For given a, p this implies the existence of a phase transition for a critical value of U

$$U_c = \frac{1}{\sqrt{2\alpha(D-2)(D-p-4)-1}}, \quad (5.32)$$

¹¹For the solutions with a z -dependence, we have used the numerical data in [35], [39].

¹²The thermodynamics of uniform black strings and p -branes has been discussed in [40] (see also [41] and references therein).

where $C_{Q,L}$ changes the sign. All solutions with $U > U_c$ (thus with a large enough electric charge) are stable in a canonical ensemble. Also, one can easily show that $C_{Q,L}$ is always negative for large enough values of a (this property is shared by the solutions with a dependence on the extra-dimension z).

This issue is especially interesting in connection with the Gubser-Mitra (GM) conjecture [42], that correlates the dynamical and thermodynamical stability for systems with translational symmetry and infinite extent. According to this conjecture, the appearance of a negative specific heat of a black brane is related to the onset of a classical instability. Supposing that this conjecture holds also for the charged solutions discussed in this paper, one would expect all charged, nonextremal black branes to be unstable, irrespective of the value of the electric charge¹³. This is indeed the case when considering charged solutions in a grand canonical ensemble. This agrees also with the results based on the map (5.13). As shown in [31], the vacuum uniform solutions with a given L are unstable below a critical mass $M_c^{(0)}$. A consequence of the Harrison transformation is that the unstable mode of the vacuum solutions would be present also for the nonextremal charged solutions. Thus, for a given U , one expects a branch of charged, nonuniform solutions to emerge for a critical value of the mass of the uniform solution $M_c = M_c^{(0)}(1 + \frac{2(D-2)(D-p-4)}{(D-p-3)(1-U^2)}U^2\alpha)$.

5.4 The $p = 0$ case: charged black strings and caged black holes

To simplify the general picture, we shall analyse for the rest of this section the simplest case of charged nonuniform black strings and black holes¹⁴ in $D = 5, 6$ KK theory, *i.e.* with a single extra-direction, $p = 0$. As we shall see, the solutions have a rich structure already in this case.

Starting with the geometrical properties, we note that for the approach in this work, the charged BHs and the NUBSs are described by the ansatz (4.1) (we recall that $f(r) = 1 - (r_0/r)^{d-4}$, $R(r,z) = r$ for black strings, while $f(r) = 1$ and $R(r,z)$ has a complicated expression for caged BHs, see Appendix A). As seen from (5.6), the inclusion of an electric charge changes the shape of the functions A, B, C .

In Figure 1 we exhibit the metric functions $-g_{tt}/f = e^{2A}$, $g_{zz} = e^{2B}$, and $g_{\theta\theta}/r^2 = e^{2C}$ for a typical $D = 6$ vacuum NUBS solution ($U = 0$) and a charged one with $U = 0.85$ in Einstein-Maxwell theory ($a = 0$). The nonuniformity parameter of these solutions as defined by (4.2) is $\lambda = 0.65$.

To obtain a more quantitative picture of the NUBS metric functions, we exhibit A, B , and C in Figure 2 for three values of U and two fixed values of z . The solutions there have a dilaton coupling constant $a = 1$ and $\lambda = 0.93$. The corresponding matter functions for the same configurations are shown in Figure 3. One can see that although the shape of the functions is qualitatively the same in both cases, the precise values taken by A, B, C in the region close to the horizon are very different. However, in both cases the metric functions exhibit an extremum at the horizon for $z = L/2$.

¹³The Gregory-Laflamme instability of $D = 5$ electrically charged black strings has been discussed in the recent work [43].

¹⁴These are the only cases where data is available for both the BHs and NUBSs branches.

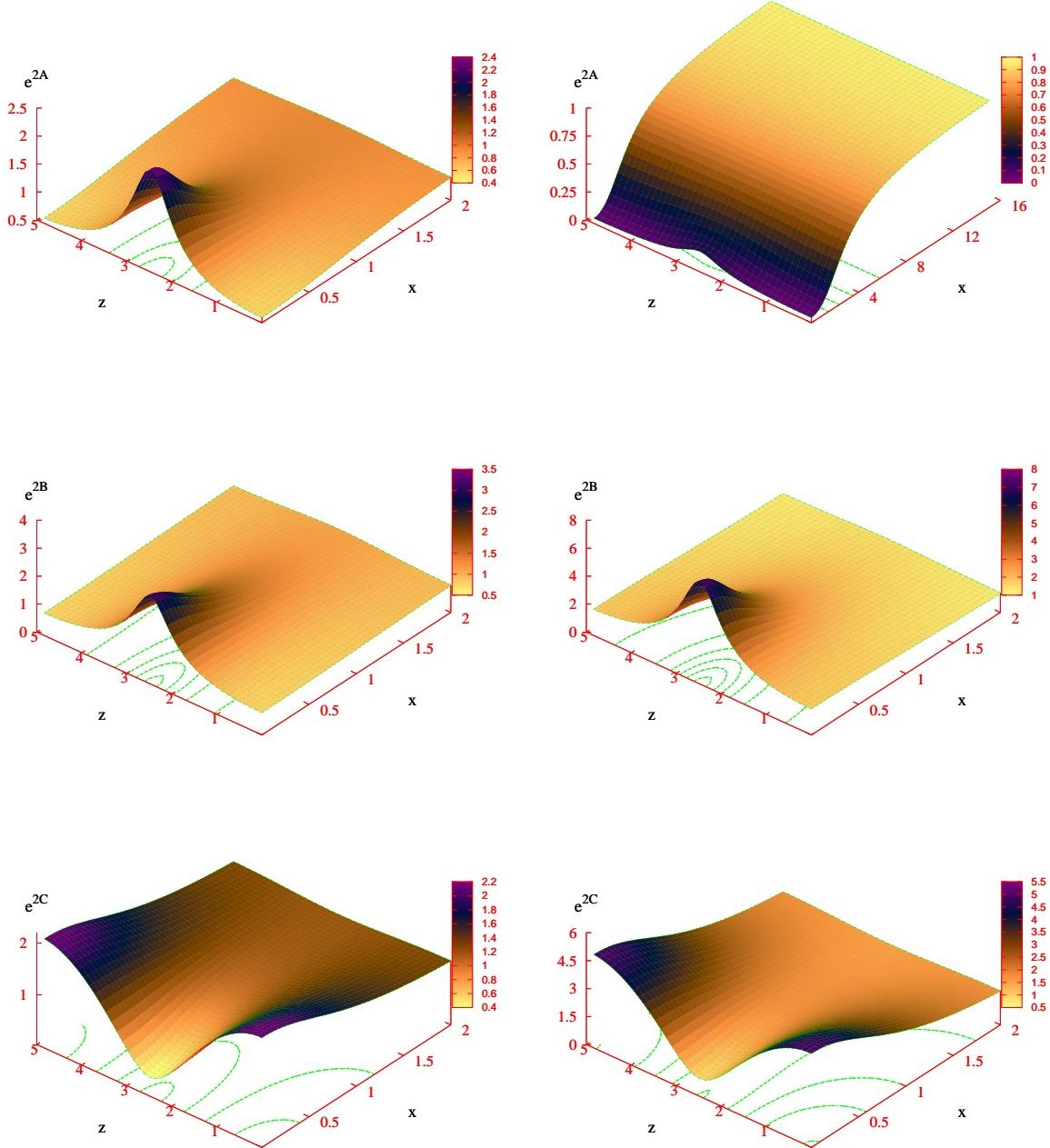


Figure 1. The metric functions e^{2A} , e^{2B} , and e^{2C} of a typical $D = 6$ nonuniform string solution are shown as functions of the coordinates $x = \sqrt{r^2 - r_0^2}, z$ for a vacuum solution (left) and a charged solution with $U = 0.85$ (right). (Note the different scale of the functions in these cases.)

The proper length of the z -circle increases with U , for any finite value of U . (The input parameters of the solutions in Figures 1-3 are $r_0 = 1, L = 4.9516$).

An interesting point here concerns the deformation of the horizon of the NUBS solu-

tions. One can see from (5.5) that although the charged solutions have different values of R_{max} , R_{min} from the vacuum case, the nonuniformity parameter λ as defined in (4.2) does not depend on the value of the chemical potential (or of the electric charge).

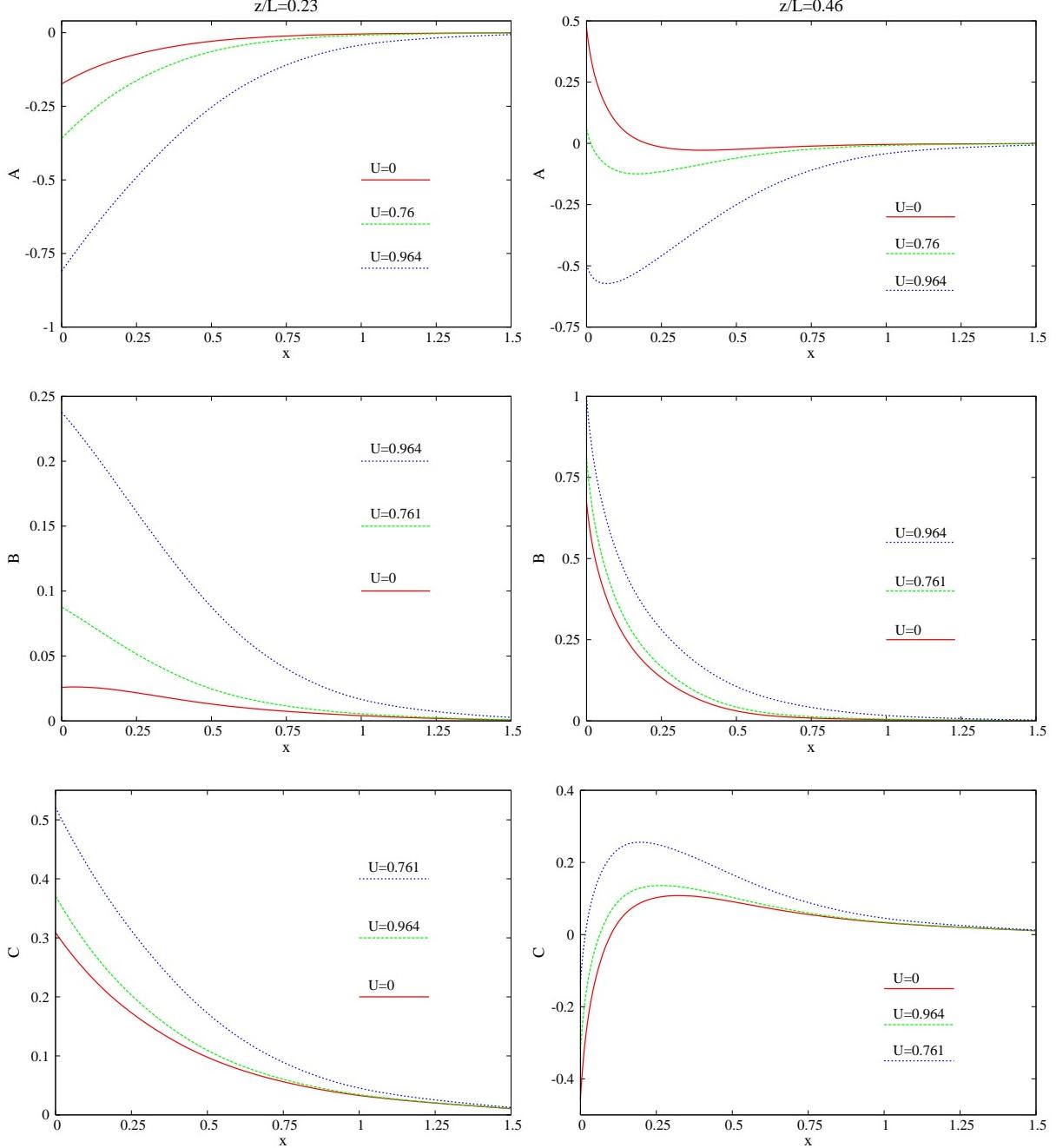


Figure 2. The metric functions A , B , and C of $D = 6$ charged nonuniform black string solutions are shown as functions of the radial coordinate $x = \sqrt{r^2 - r_0^2}$, for three values of the parameter U .

Therefore, the qualitative features of the horizon geometry remain the same as in the vacuum case. In particular, the parameter λ approaches very large values for solutions constructed using seed NUBS close to the transition point (this occurs for any value of U).

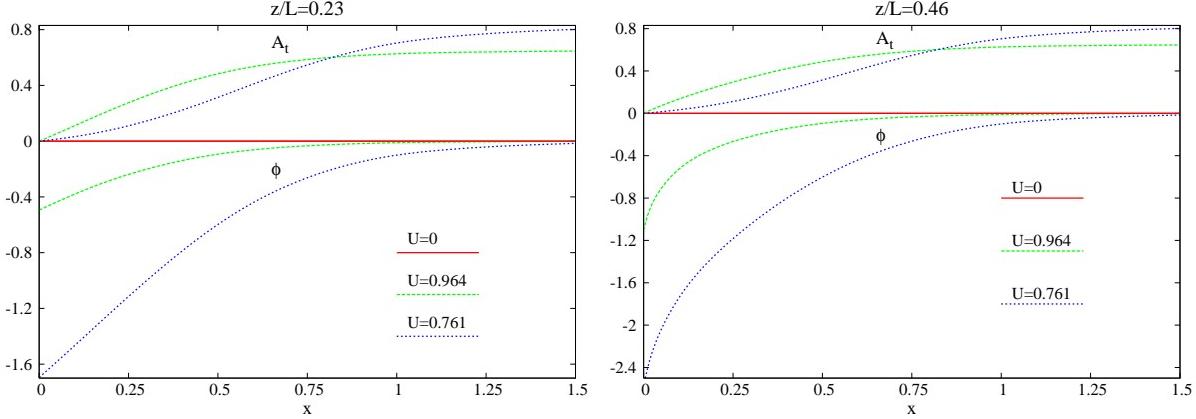


Figure 3. The dilaton ϕ and the electric potential A_t are shown for the same solutions exhibited in Figure 2.

The geometrical properties of the caged BH solutions can be discussed in a similar way. In Figure 4 we exhibit the metric functions e^{2A} , e^{-2B} and e^{2C} for a typical $D = 6$ vacuum BH solution ($U = 0$) and a charged one with $U = 0.85$ in EM theory ($a = 0$). The position of the event horizon there was $r_0 = -1$ (see Appendix A for more details on the seed solutions). As a generic feature, for any value of the parameter U , the ratio between R_{eq} and L_{axis} remains the same as in the vacuum case. At a technical level, this is a consequence of the fact that the Harrison transformation (2.5) involves just a conformal transformation of the spacelike part $g_{\alpha\beta}dx^\alpha dx^\beta$ of the seed geometry.

This observation is relevant in connection with the issue of a topology changing transition between the black hole and the NUBS branches. Since, from (5.13), adding an electric charge implies just a recombination of the global charges of the system, we argue that the picture one finds here would be qualitatively similar to that valid for vacuum solutions.

To clarify this point, we show in Figure 5 the mass M and temperature T versus the relative string tension n , for the nonuniform string branch as well as for the black hole branch¹⁵, for several values of the parameter U . The black hole data are taken from [35]. (For convenience, we normalize in this subsection the thermodynamic quantities by their values for the corresponding charged UBS with the same value of U .)

¹⁵The same picture is found when considering the (n, M) diagram in $D = 5$ and the (n, T) plot in $D = 6$, respectively. For all values of U , we note qualitative agreement of the shape and the relative position of the charged nonuniform string branch and the charged black hole branch in five dimensions with the shape and relative position of the corresponding branches in six dimensions. The backbending noticed in [39] for the vacuum case is present also for charged NUBS, both in five and in six dimensions.

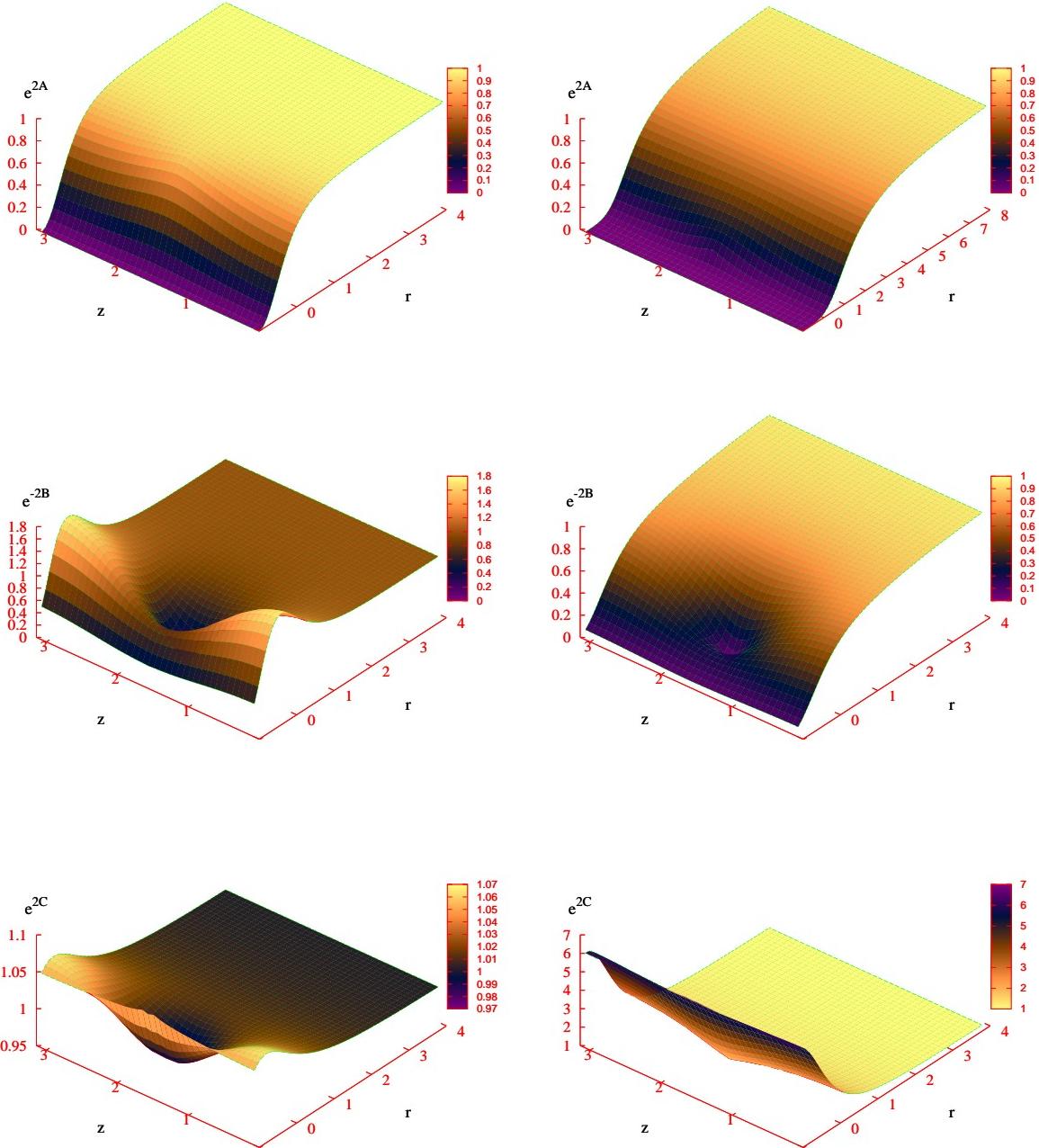


Figure 4. The metric functions e^{2A} , e^{-2B} , and e^{2C} of a typical $D = 6$ KK black hole solution are shown as functions of the coordinates r, z for a vacuum solution (left) and a charged solution with $U = 0.964$ (right).

One can see that we have all reasons to expect that the topology changing transition scenario proposed in the vacuum case, holds here also. However, as seen in Figure 5, a nonzero electric charge translates into a different position in the (\bar{M}, \bar{n}) plane (and (\bar{T}_H, \bar{n}) ,

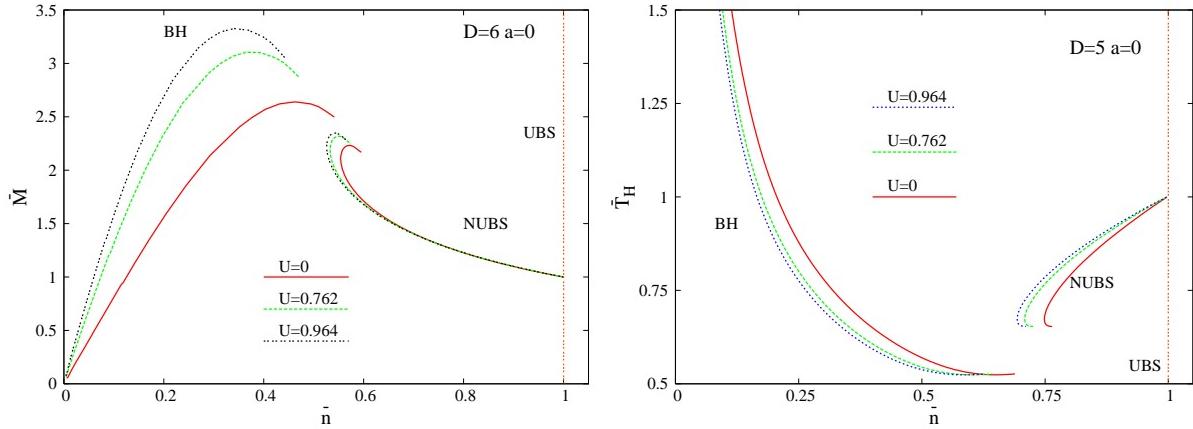


Figure 5. *Left:* The mass M of the $D = 6$ charged nonuniform string and black hole branches is shown versus the relative string tension n . *Right:* The Hawking temperature T_H of the $D = 5$ charged nonuniform string and black hole branches is shown versus the relative string tension n (these plots are given for three values of U). \bar{M} , \bar{n} and \bar{T}_H correspond to the quantities normalized by the corresponding values of the uniform string solutions with the same values of U .

respectively) of the *putative* transition point. We have also noticed that as U increases towards its maximal value, the $\bar{M}(\bar{n})$ curves converge to a critical one, which is close to that corresponding to $U = 0.964$ in Figure 5. The same holds for the (\bar{n}, \bar{T}_H) diagram.

The general thermodynamical properties of the charged solutions are rather complicated as compared to the vacuum case, with a nontrivial dependence on the dilaton coupling constant a . However, the qualitative features are again the same for both $D = 5$ and $D = 6$ solutions.

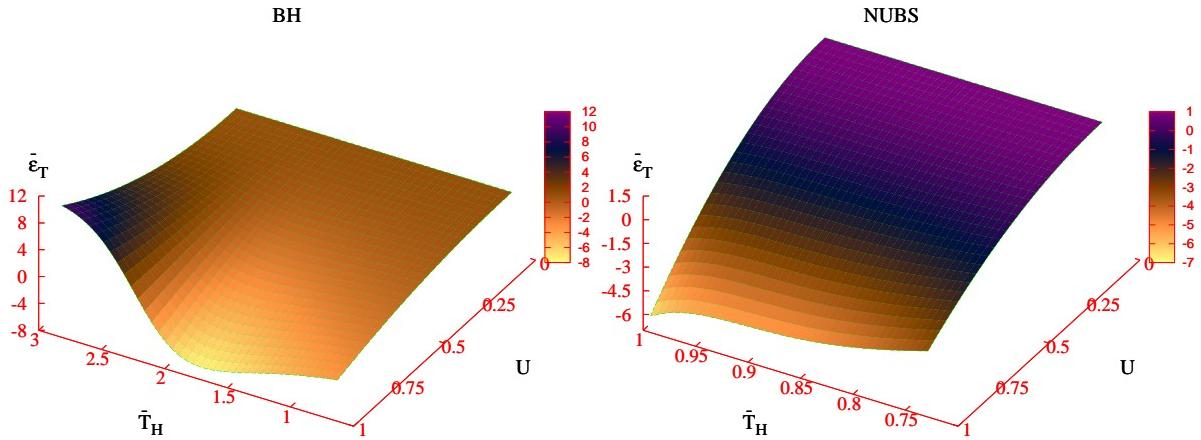


Figure 6. The scaled isothermal electric permittivity is shown as a function of the normalized temperature and the parameter U for the $D = 6$ charged caged black holes (left) and nonuniform black string strings (right).

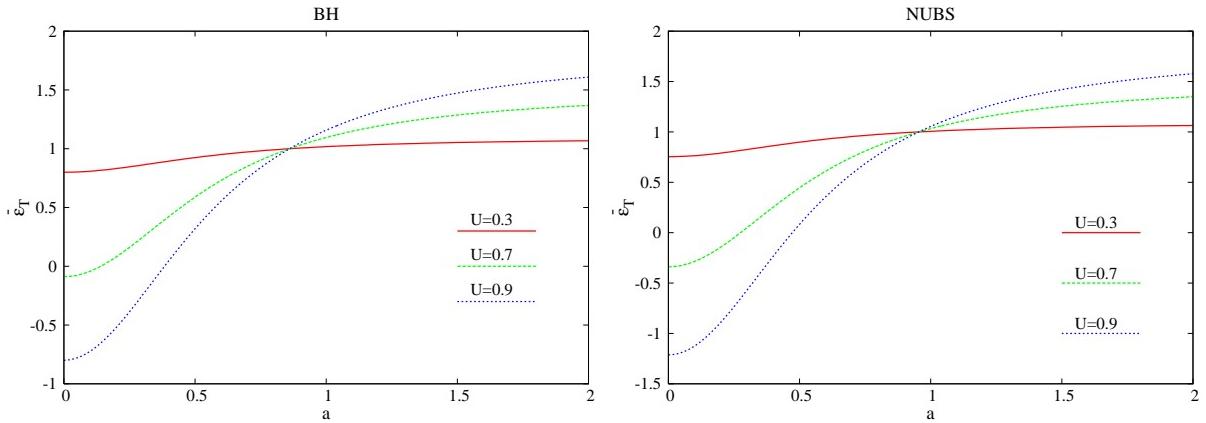


Figure 7. The scaled isothermal electric permitivity is shown as a function of the dilaton coupling constant a for the $D = 5$ charged caged black holes (left) and nonuniform black strings (right). The plots here correspond to a fixed normalized temperature $\bar{T}_H = 0.7$ and several values of the chemical potential U .

Also, as argued above, both the BHs and the NUBSs are thermodynamically unstable in a grand canonical ensemble¹⁶. Here we start by showing in Figure 6 the scaled isothermal electric permitivity¹⁷

$$\bar{\epsilon}_T = U^2 + 1 + 2\alpha(D-2)U^2\left(1 + \frac{C_{Q,L}^0}{S_0}\right) \quad (5.33)$$

for caged BHs and NUBS solutions as a function of the normalized temperature and the parameter U . These plots correspond to a value $a = 0$ of the dilaton coupling constant, *i.e.* the EM theory. One can see that, for NUBS, ϵ_T is a decreasing function of U for any allowed value of the temperature. However, it becomes negative for large enough values of the chemical potential. The NUBS solutions close to the transition point have a positive ϵ_T for small U only. As seen in Figure 6 (left), the picture is more complicated for caged BHs. The small charged black holes (*i.e.* with large values of T_H) have always a positive ϵ_T . However, larger black holes have a negative isothermal electric permitivity, for large enough U .

This picture depends however on the value of the dilaton coupling constant a . Our study indicates that for both types of black objects ϵ_T becomes positive for large enough values of a , irrespective of the values of U, T_H (see Figure 7).

¹⁶As discussed in [39], an unexpected feature of the NUBS solutions is the occurrence of a new branch at a critical (minimal) value of the relative string tension, close to transition point. Along this branch (which has a small extension in the (M, n) -plane, see Figure 5), the mass decreases with n , while the specific heat is still negative. In what follows, to simplify the general picture, we shall ignore this secondary branch for all plots of the specific heat and isothermal electric permitivity. The same holds for caged BH solutions, the last points for the data in [35] indicating the existence of a (small) secondary branch close to the transition point, see Figure 5 (this feature was confirmed by the recent results in [34]). This secondary BH branch was also ignored when discussing $C_{Q,L}$ and ϵ_T .

¹⁷ $\bar{\epsilon}_T$ has the same sign as the isothermal electric permitivity and stays finite as $U \rightarrow 1$.

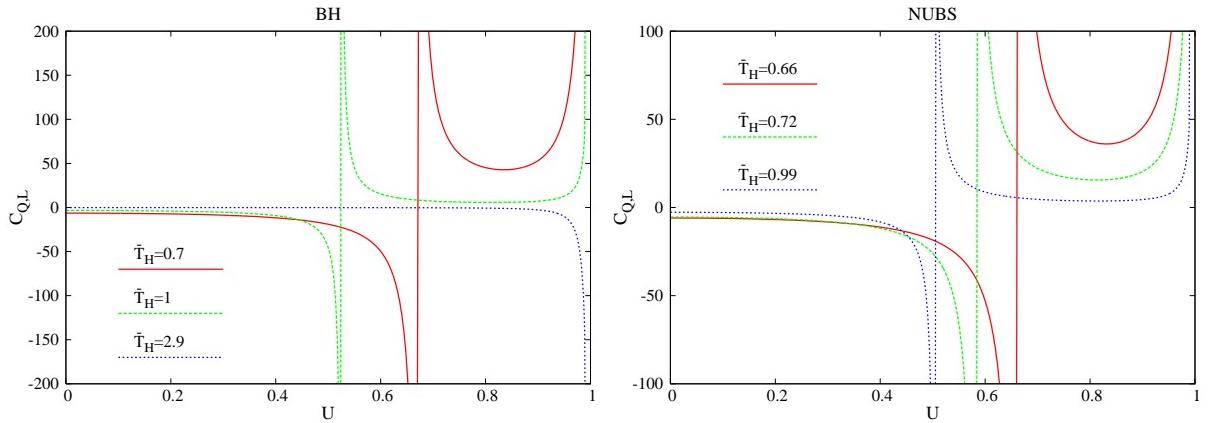


Figure 8. The specific heat is shown as a function of U for the $D = 5$ charged caged black holes (left) and nonuniform black string strings (right). The plots here correspond to a vanishing dilaton coupling constant $a = 0$ and several values of the normalized temperature.

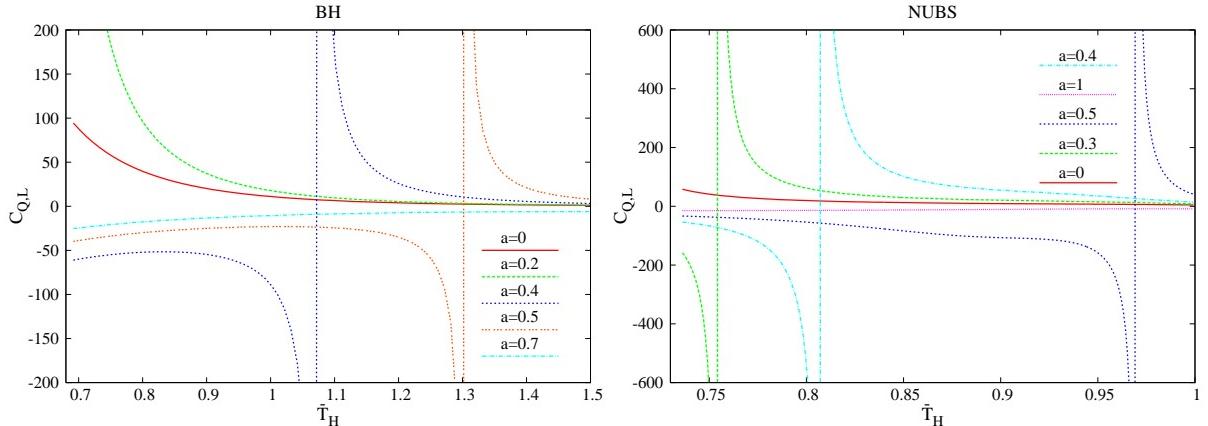


Figure 9. The specific heat is shown as a function of the scaled temperature for the $D = 6$ charged caged black holes (left) and nonuniform black strings (right). These plots are for a fixed value of the chemical potential $U = 0.5$ and several values of the dilaton coupling constant.

This can be understood by observing that for large a , the term $U^2 + 1$ becomes dominant in the numerator of the isothermal electric permitivity.

In Figure 8 we present the specific heat as a function of the parameter U for several values of the rescaled temperature for black strings and caged black holes in $D = 5$ EM theory (a similar picture is found in $D = 6$). The specific heat of the black strings is always negative for small values of U . However, for any T_H , it changes the behaviour for a critical value of the chemical potential $U = U_c$ (which is fixed by the condition $\epsilon_T(U_c) = 0$). $C_{Q,L}$ exhibits a discontinuity at U_c , indicating the existence of a phase transition in the thermodynamical system. The value of U_c decreases as the scaled temperature increases.

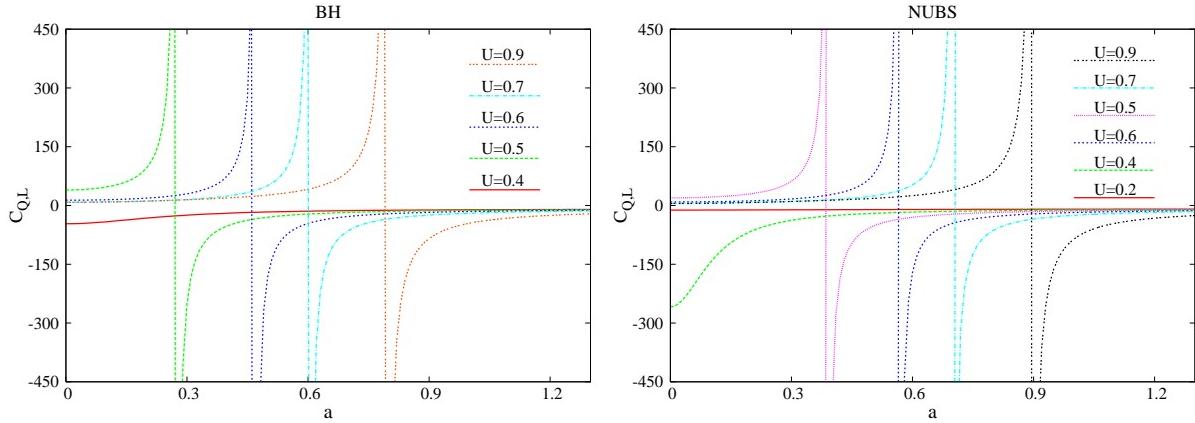


Figure 10. The specific heat is shown as a function of the dilaton coupling constant for the $D = 6$ charged caged black holes (left) and nonuniform black strings (right). The plots here correspond to a fixed value of the scaled temperature $\bar{T}_H = 0.8$ and several values of the chemical potential.

The picture is slightly different for caged black holes. The scenario we found for NUBSs applies only for solutions close enough to the critical region where the topology changing transition is expected to occur. The solutions generated from small caged BHs with a large enough scaled temperature have a negative specific heat (see Figure 8 left).

Again this picture depends on the value of the dilaton coupling constant. In Figure 9, the specific heat is shown as a function of the scaled temperature for $D = 6$ solutions with a fixed value of U and a range of a .

The dependence of the solutions on the dilaton coupling constant a is further illustrated in Figure 10, where we show the specific heat as a function of a for several values of U . For small values of a , one notices the possible existence of configurations with a positive specific heat, for both caged BHs and NUBSs. One can see that for large enough values of the dilaton coupling constant, the specific heat becomes negative irrespective of the value of U , for both black holes and black strings. This can be understood by noticing that, since $\alpha \rightarrow 0$ for large enough a , the sign of the specific heat (5.24) of the charged solutions is fixed by $C_{Q,L}^0$, which is a negative quantity.

6. Slowly rotating charged NUBS

It would be interesting to generalize the charged KK configurations discussed in the previous sections by adding a nonvanishing angular momentum. Unfortunately, there is no simple procedure to spin a given static system. Moreover, all known techniques fail for the case of KK solutions with a dependence on the extra-dimension. Some of the reasons for this are that the great majority of the solution generating techniques rely heavily either on high degrees of symmetry in the metric (such as the geometries in the generalized Weyl class) or on the possibility of dimensional reduction down to 3-dimensions where target

space symmetries can be used to add rotation to a given static solution. As we have previously stated, solutions with a dependence on the extra-dimension clearly do not belong to these classes of geometries and therefore any rotating non-uniform solution has to be found by brute force solving of the field equations.

To the best of our knowledge, the only rotating solutions with a dependence on the extra-dimension correspond to the NUBSs discussed in [44]. The solutions there have been constructed in $D = 6$ dimensions, for the special case of two equal magnitude angular momenta. The angular dependence factorizes in this case, and the problem reduces to solving a set of partial differential equations with dependence only on the radial variable r and the extra-dimension variable z . $D = 6$ charged nonuniform black string configurations in heterotic string theory were also constructed in [44] by employing a solution-generating technique. It would be interesting to extend these results for other values of D and non-equal angular momenta. The case of a black string with a single angular momenta is of particular interest, given the complicated picture one finds already for $D > 4$ asymptotically Minkowski black hole solutions [45]. However, this leads to a difficult numerical problem, since the equations would depend on three variables. Moreover, as far as we know, rotating EM-dilaton black strings do not exist in closed form even for the uniform case¹⁸, and one has to rely on numerical methods to construct them.

As an extension of the results in the previous sections, we present in what follows a numerical study of the NUBS solutions with a single angular momentum, in the limit of slow rotation (the case of UBSs is discussed in Appendix B, where a closed form solution is presented). For such configurations, the equations depend only on the radial variable r and the extra-dimension variable z and are solved to first order in the angular momentum parameter. To simplify the general picture, we shall restrict here to the case of EM theory, although the results can easily be generalized to include a dilaton field.

In this approach, one starts with the following general ansatz depending on r, z and the angular variable θ , describing charged rotating NUBS solutions with a single angular momentum:

$$ds^2 = -e^{2\bar{A}(r,z,\theta)} f(r) dt^2 + e^{2\bar{B}(r,z,\theta)} \left(\frac{dr^2}{f(r)} + e^{2S(r,z,\theta)} dz^2 \right) + e^{2\bar{C}(r,z,\theta)} r^2 \left(d\theta^2 + e^{2F(r,z,\theta)} \sin^2 \theta (d\varphi + J(r,z,\theta) dt)^2 + e^{2H(r,z,\theta)} \cos^2 \theta d\omega_{D-5}^2 \right),$$

$$\mathcal{A} = A_\mu(r,z,\theta) dx^\mu, \quad (6.1)$$

(we recall that $f(r) = 1 - (r_0/r)^{D-4}$ in this case). Then we suppose the following expansion of the functions in (6.1) in terms of a small parameter a (which should not be confused with the dilaton coupling constant):

$$X_i(r,z,\theta) = X_i^{(0)}(r,z) + a X_i^{(1)}(r,z,\theta),$$

with $X_i^{(0)}(r,z)$ being the functions which enter the static solution (e.g. $\bar{A}(r,z,\theta) = A(r,z) + a A_1(r,z,\theta)$, $\bar{B}(r,z,\theta) = B(r,z) + a B_1(r,z,\theta)$, $C(r,z,\theta) = C(r,z) + a C_1(r,z,\theta)$ etc.).

¹⁸Except for special values of the dilaton coupling constant where the solutions can be generated from the Myers-Perry black holes following the techniques in [46], [47].

A straightforward but cumbersome computation shows that for infinitesimal a , the only term in the metric that changes is $g_{t\varphi}$ and we find the following consistent metric form:

$$ds^2 = -e^{2A(r,z)} f(r) dt^2 + e^{2B(r,z)} \left(\frac{dr^2}{f(r)} + dz^2 \right) + e^{2C(r,z)} r^2 (d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\omega_{D-5}^2) - 2a \sin^2 \theta j(r,z) d\varphi dt, \quad (6.2)$$

the angular dependence of the nondiagonal component being factorized. Similarly, the only component of the vector potential that changes is A_φ , the corresponding expression of the Maxwell field being

$$\mathcal{A} = A_t(r,z) dt + a \sin^2 \theta h(r,z) d\varphi. \quad (6.3)$$

Thus the problem is reduced to solving the equations for $j(r,z), h(r,z)$ in the background of the static charged solutions. These equations read (to simplify the relations, we denote here $A_t = V$):

$$\begin{aligned} j_{,rr} + \frac{j_{,zz}}{f} - (V_{,r} h_{,r} + \frac{V_{,z} h_{,z}}{f}) - (A_{,r} j_{,r} + \frac{A_{,z} j_{,z}}{f}) + 2j(A_{,r} C_{,r} + \frac{A_{,z} C_{,z}}{f}) \\ + (D-5)(C_{,r} j_{,r} + \frac{C_{,z} j_{,z}}{f}) - 2j(C_{,rr} + \frac{C_{,zz}}{f}) - 2(D-3)j(C_{,r}^2 + \frac{C_{,z}^2}{f}) \\ + \frac{2j}{r}(A_{,r} - 2(D-3)C_{,r} - \frac{D-4}{r}) + \frac{(D-5)j_{,r}}{r} = 0, \end{aligned} \quad (6.4)$$

$$\begin{aligned} h_{,rr} + \frac{h_{,zz}}{f} + (A_{,r} h_{,r} + \frac{A_{,z} h_{,z}}{f}) + (D-5)(C_{,r} h_{,r} + \frac{C_{,z} h_{,z}}{f}) + \frac{e^{-2A}}{f}(2j(V_{,r} C_{,r} + \frac{V_{,z} C_{,z}}{f}) \\ - (V_{,r} j_{,r} + \frac{V_{,z} j_{,z}}{f})) + \frac{2e^{-2A}j}{rf} V_{,r} + (\frac{D-5}{r} + \frac{f'}{f}) h_{,r} - \frac{2(D-4)e^{2B-2C}}{r^2 f} h = 0. \end{aligned} \quad (6.5)$$

The equation for j has a total derivative structure which helps us relate the total angular momentum to event horizon quantities:

$$\frac{1}{L} \int_0^L dz e^{-A+(D-3)C} r^{D-3} \left(j_{,r} - \frac{2j}{r} - 2jC_{,r} - hV_{,r} \right) \Big|_{r=r_h} = \lim_{r \rightarrow \infty} r^{D-3} \left(j_{,r} - \frac{2j}{r} \right). \quad (6.6)$$

For the slowly rotating solutions, the horizon radius is still determined by the equation $f(r_0) = 0$. Also, the surface gravity and area of the event horizon do not change to order $\mathcal{O}(a)$. However, the angular momentum J and the magnetic dipole moment are evinced already at this order. Namely, the angular momentum is related to the asymptotic form of the metric by

$$g_{t\varphi} = -\frac{J}{2\Omega_{D-3} L} \frac{\sin^2 \theta}{r^{D-4}}, \quad (6.7)$$

while the magnetic dipole moment μ is read from the asymptotic form of the magnetic gauge potential $h(r,z)$

$$h(r,z) = -\frac{\mu}{(D-4)\Omega_{D-3} L} \frac{1}{r^{D-4}}. \quad (6.8)$$

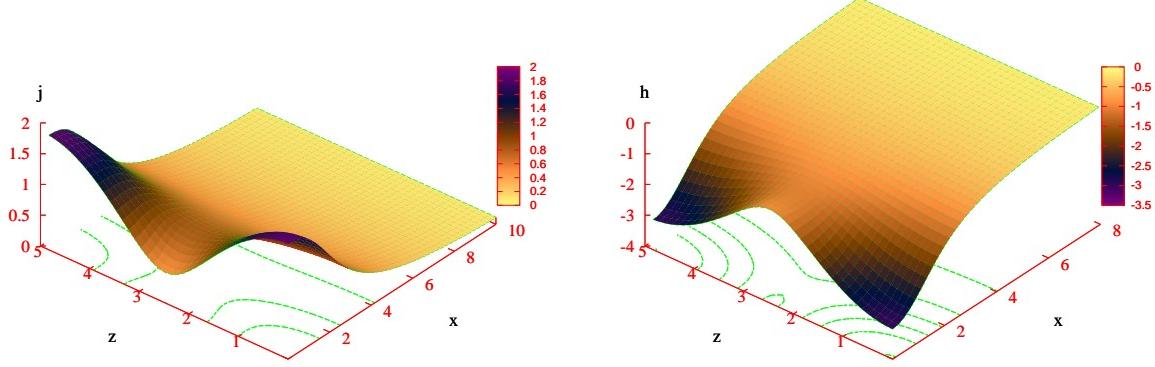


Figure 11. The functions $j(r,z)$ and $h(r,z)$ are shown for a typical $D = 6$ slowly rotating NUBS in Einstein-Maxwell theory (with $x = \sqrt{r^2 - r_0^2}$). The parameters of this configuration are $r_0 = 1, L = 4.9516, \lambda = 0.47, U = 0.833$.

In practice, the idea is to start with a known vacuum NUBS solution and to charge it by using the Harrison transformation described in the previous sections. This generates static charged NUBS solutions of the EM equations (*i.e.* $a = 0$, no dilaton field). Then the eqs. (6.5) are solved with the background functions A, B, C .

We have performed this computation for $D = 6$ NUBS solutions¹⁹ with several values of the nonuniformity parameter λ (with λ defined by the equation (4.2)) and a range of U . The equations (6.4) were solved by using the same methods as those described in Appendix A for the vacuum NUBS case. In this scheme, one introduces a new radial coordinate $x = \sqrt{r^2 - r_0^2}$ such that the event horizon is located at $x = 0$. Then the equations (6.5) are solved subject to the following set of boundary conditions:

$$j|_{x=0} = j_H e^{2C(0,z)}, \quad \partial_x h|_{x=0} = 0, \quad j|_{x=\infty} = h|_{x=\infty} = 0, \quad \partial_z j|_{z=0,L} = \partial_z h|_{z=0,L} = 0, \quad (6.9)$$

with j_H a constant fixing the angular velocity of the horizon, $\Omega_H = aj_H/r_0^2$.

The profiles of the functions J, h for a typical solution are presented in Figure 11. One can see that they present a strong dependence on the extra-coordinate z , possessing a peak around $z = L/2$.

A quantity of interest here is the gyromagnetic ratio g which is defined in the usual way by

$$g = \frac{2M\mu}{Q_e J}. \quad (6.10)$$

In $D = 4$ classical electrodynamics, the gyromagnetic ratio of a charged rotating body is always $g = 1$. However, the gyromagnetic ratio of an electron in Dirac theory is $g = 2$, which

¹⁹We expect slowly rotating NUBS solutions to exist for any value of D . However, the case $D = 6$ is more convenient for numerics when dealing with KK solutions with a dependence on the extra-dimension, see *e.g.* the comments on that in [48], [33].

is precisely the value found for a rotating Kerr-Newmann black hole. When considering the higher dimensional version of these EM black holes²⁰, the lowest order perturbation theory gives for the gyromagnetic ratio the result $g = D - 2$, which seems a natural higher dimensional generalization of the gyromagnetic ratio in $D = 4$ dimensions [49], [50]. However, numerical calculations revealed that in higher dimensions, in a nonperturbative approach, the gyromagnetic ratio is not constant, but deviates from $D - 2$ for finite values of the charge [51], [52], which was also confirmed by higher order perturbation theory [53].

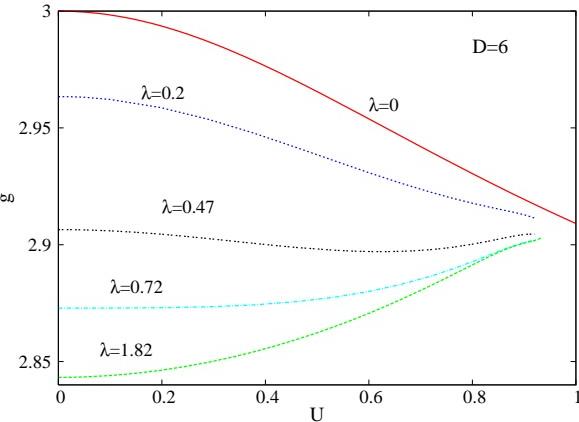


Figure 12. The gyromagnetic ratio g is shown as a function of U for several values of the nonuniformity parameter λ for $D = 6$ slowly rotating black string solutions in Einstein-Maxwell theory.

However, so far most of the studies in the literature were limited to the case of black holes in asymptotically flat spacetime, the gyromagnetic ratio of solutions with compact extra-dimensions being not explored. As found in Appendix B, g varies for uniform black string solutions between $D - 3$ (no charge, $U = 0$) and $\frac{(D-2)^2(D-4)}{D(D-5)+5}$ (in the extremal limit, $U \rightarrow 1$). One can see in Figure 12 that the picture is much more complicated for $D = 6$ solutions with an explicit dependence on the extra-coordinate. For small values of λ , the gyromagnetic ratio g still decreases with U . However, the situation changes for large enough values of the nonuniformity parameter λ . Our numerical results suggest that, for any λ , in the extremal limit, the gyromagnetic ratio of NUBS solutions approaches the same value $g_c = 32/11$.

It would be interesting to perform a similar computation for caged black holes and to consider the issue of topology change for spinning solutions.

7. Conclusions

The charged solutions in string theory and M-theory are usually obtained from the vacuum KK black objects in d -dimensions by sequences of boosts and dualities [8]. This gave a precise connection between the rich phase structure of KK black objects and that of the

²⁰Note that the $D > 4$ counterparts of the Kerr-Newmann black hole are not known in closed form.

corresponding branes in string/M-theory and provided predictions for the thermodynamics of supersymmetric Yang-Mills theories on a circle. Our proposal here was to use instead the Harrison transformation derived in [7]. This construction is more general since it holds for any value of the dilaton coupling constant a and any value of p (the brane extra-dimensions), in particular for the Einstein-Maxwell (EM) theory ($p = 0$).

The main purpose of this paper was to use this technique in order to investigate the issue of charged black objects in KK theory for an arbitrary value of a , by using the known vacuum NUBS and the KK localized BHs as seed solutions. We have observed that both the geometrical and thermodynamical properties of the charged solutions can be derived from the corresponding D -dimensional vacuum configurations. Asymptotically flat black objects generated in this way from vacuum fields have a geometrical structure similar to that of the vacuum seed solutions.

A natural question that arises at this point is how to compute the various conserved quantities and the total action of the charged black objects. One particularly powerful approach to this problem is given by the counterterms method. For asymptotically flat spaces there have been proposed many such counterterms and their form relies heavily on the topology at infinity [14], [25]-[27]. However, in certain situations, these counterterms fail to regularize the gravitational action. One recent proposal put forward by Mann and Marolf [9] seems to avoid these problems for asymptotically flat spaces. A secondary aim of this paper was to propose a direct extension of the Mann-Marolf counterterm to ‘*transverse asymptotically flat*’ spaces in KK theory. In Section 3 we showed that the total action and conserved charges can be expressed in a novel way in terms of the electric part of the Weyl tensor and that, moreover, the final results are consistent with those previously derived in the p -brane literature.

The thermodynamics of charged black objects in KK theory has been considered in Section 5. There we have argued that these solutions are thermodynamically unstable in a grand canonical ensemble. Also, our results show that the thermodynamics of these black objects is affected by the value of the dilaton coupling constant. A general discussion of the charged black strings and black holes for the simplest case of a KK theory in $D = 5, 6$ dimensions with a single compact extra-dimension has also been studied there. These solutions display a rich phase-structure already in this case, some of them being thermally stable in a canonical ensemble, for large enough values of the electric charge.

The charged solutions we have discussed are likely to possess spinning generalization. Unfortunately, the Harrison transformation considered in our work is restricted to the case of static configurations. In Section 6 we have presented a numerical study of the uniform and NUBS solutions with a single angular momentum, in the limit of slow rotation. Our results provide evidence for the existence of such solutions and show that the gyromagnetic ratio depends on the nonuniformity parameter of the black strings.

Although we have restricted our study to the (physically more relevant) region outside the horizon, it would be interesting to consider also the interior of these charged black objects. In the vacuum case, the ref. [54] presents numerical evidence, that the singularity touches the horizon as the horizon topology changing transition is reached. We expect that the inclusion of the electric charge would lead to a more complicated picture, at least in

the absence of a dilaton field.

In this paper we have considered the simplest case of a single gauge field. In principle, one can easily add secondary gauge fields to the action (2.1) by considering sequences of boosts and dualities (this would also generate supplementary scalars, with a specific coupling to the gauge fields). The methods in this work can easily be extended to that case. On general grounds, we expect the features we have found to be generic; in particular, all solutions are likely to be unstable in a grand canonical ensemble.

Also, as an avenue for future research, it would be interesting to study KK solutions with more general matter fields than the model (2.1). The case of configurations with nonabelian gauge fields is of particular interest, given the rich structure of these systems and the interplay in this case between internal group symmetries and the spacetime symmetries [55].

Acknowledgements

We are grateful to T. Wiseman for kindly providing us with the $D = 5$ and $D = 6$ black hole data in [35]. BK gratefully acknowledges support by the DFG. The work of ER was supported by a fellowship from the Alexander von Humboldt Foundation. The work of CS was supported by the Natural Sciences and Engineering Council of Canada.

A. The construction of vacuum Kaluza-Klein solutions: technical aspects

A.1 NUBS solutions

The NUBS solutions were constructed by several different groups [35], [39], [48], [56], with different numerical methods, and are known for all dimensions up to eleven.

The basic steps in the nonperturbative construction of these solutions are as follows. First, the numerics is performed by using a new radial coordinate x , which is defined such that the r, z part of the metric becomes conformally flat²¹, *i.e.*

$$x = \int \frac{dr}{\sqrt{f(r)}} + x_0, \quad (\text{A.1})$$

and

$$ds_d^2 = -e^{2A(x,z)} f(r(x)) dt^2 + e^{2B(x,z)} (dx^2 + dz^2) + e^{2C(x,z)} r^2(x) d\omega_{d-3}^2. \quad (\text{A.2})$$

The constant x_0 in the above relation is fixed by imposing that the horizon is located at $x = 0$, where $f(r_0) = 0$ (moreover, $x/r \rightarrow 1$ in the asymptotic region).

The Einstein equations $G_t^t = 0$, $G_x^x + G_z^z = 0$ and $G_\theta^\theta = 0$ (where θ denotes an angle of the $D - 3$ dimensional sphere and G_a^b the Einstein tensor) yield three elliptic equations for the metric functions A , B , C (these equations are given *e.g.* in Section 2 of ref. [39]). The remaining Einstein equations $G_z^x = 0$, $G_x^x - G_z^z = 0$ yield two constraints. As first observed in [48], these constraints satisfy Cauchy-Riemann relations

$$\partial_z (\sqrt{-g} G_z^x) + \partial_x (\sqrt{-g} \frac{1}{2} (G_x^x - G_z^z)) = 0, \quad \partial_x (\sqrt{-g} G_z^x) - \partial_z (\sqrt{-g} \frac{1}{2} (G_x^x - G_z^z)) = 0. \quad (\text{A.3})$$

²¹For example, in $D = 6$, $x = \sqrt{r^2 - r_0^2}$.

Thus the weighted constraints satisfy Laplace equations, and they are fulfilled, when one of them is satisfied on the boundary and the other at a single point [48].

All known KK solutions with a dependence on the extra-dimension have the reflection symmetry w.r.t. $z = L/2$. Thus, in numerics it is enough to consider only half of the z -domain.

The NUBS solutions are then constructed subject to the following set of boundary conditions

$$\begin{aligned} A|_{x=\infty} &= B|_{x=\infty} = C|_{x=\infty} = 0, \\ B|_{x=0} - A|_{x=0} &= d_0, \quad \partial_x A|_{x=0} = \partial_x C|_{x=0} = 0, \\ \partial_z A|_{z=0,L/2} &= \partial_z B|_{z=0,L/2} = \partial_z C|_{z=0,L/2} = 0, \end{aligned} \quad (\text{A.4})$$

where the constant $d_0 > 0$ is related to the Hawking temperature of the solutions. Regularity further requires that the condition $\partial_x B|_{x=0} = 0$ holds for the solutions.

The Hawking temperature and entropy of the NUBS solutions are given by

$$T_H = e^{A_0 - B_0} T_H^0, \quad S = S_0 \frac{1}{L} \int_0^L e^{B_0 + (D-3)C_0} dz, \quad (\text{A.5})$$

where T_H^0 , S_0 are the corresponding quantities of the uniform solution with $A = B = C = 0$

$$T_H^0 = \frac{D-4}{4\pi r_0}, \quad S_0 = \frac{1}{4G} L \Omega_{D-3} r_0^{D-3}, \quad (\text{A.6})$$

and $A_0(z)$, $B_0(z)$, $C_0(z)$ are the values of the metric functions on the event horizon $r = r_0$ ($x = 0$). The mass and tension of these solutions are given by (3.29), where the constants c_t , c_z are read from the asymptotics of the metric functions A and B , respectively.

All known NUBS solutions were found by solving numerically the set of three coupled non-linear elliptic partial differential equations for A , B , C , subject to the above boundary conditions. For a better numerical accuracy, ref. [39] proposed to employ a new radial coordinate $\bar{r} = x/(1+x)$, which maps spatial infinity to the finite value $\bar{r} = 1$.

The branch of nonuniform strings is obtained by starting at the critical point of the uniform string branch and varying the boundary parameter d_0 , which enters the Eq. (A.4). The resulting solutions have a nontrivial dependence on z , the parameter λ as defined by the equation (4.2) increasing with d_0 for the main NUBS branch. Usually, the NUBS solutions are found with good accuracy (as verified *e.g.* by using the Smarr law (4.3)), except for the configurations close to the topology changing point (*i.e.* for very large values of the nonuniformity parameter λ).

A.2 Caged black holes

Given the mismatch between the horizon topology and the asymptotic structure of space-time²² the numerics is much more difficult in this case, the only nonperturbative results available so far in the literature being for $D = 5, 6$ dimensions [33], [35], [61]. The ansatz

²²It is interesting to notice the analogy with the $D > 4$ asymptotically Minkowski black holes with a nonspherical topology of the horizon which present the same feature. For example, the horizon topology

(4.1) corresponds to that used by Kudoh and Wiseman in [33], [35]. The independent results in [61] were found by using a more complicated approach, with two different coordinate patches: the first patch covers the region close to the horizon while the second one is used in the asymptotic region (the same approach was employed in the recent work [34]). The equations are relaxed on both patches one after the other. Moreover, the patches communicate, providing boundary data to each other in the overlapping region.

We have considered the problem of constructing KK black holes by following the approach in [33], [35]. A generalization to any D of the metric ansatz used there is given by

$$ds^2 = e^{2\bar{B}(r,z)} J(r,z) B_0^2(r) (dr^2 + dz^2) + e^{2\bar{C}(r,z)} R^2(r,z) d\omega_{D-3}^2 - e^{2\bar{A}} A_0^2(r) dt^2 , \quad (\text{A.7})$$

where

$$A_0(r) = \tanh((D-3)(r-r_0)/2), \quad B_0(r) = (1+e^{-(D-3)(r-r_0)})^{2/(D-3)}, \quad (\text{A.8})$$

and

$$\begin{aligned} J(r,z) &= \frac{e^r}{2\sqrt{\sinh^2 r + \cos^2 z}}, \\ R(r,z) &= r + \frac{1}{2} \log(1 + 2e^{-r} \sqrt{\sinh^2 r + \cos^2 z} + e^{-2r} \sqrt{-1 + (e^{2r} + 2e^r \sqrt{\sinh^2 r + \cos^2 z})^2}), \end{aligned} \quad (\text{A.9})$$

while $r_0 < 0$ is an input parameter corresponding to the event horizon radius. As it was extensively discussed in [33], r_0 also specifies the size of the black hole. Without any loss of generality, the compactification radius is fixed here to be $L = \pi$.

One can see from (A.7) that the metric functions A, B, C in the generic ansatz (4.1) contain some background terms (*e.g.* $A(r,z) = \bar{A}(r,z) + \log A_0(r)$ etc.). The functions $A_0(r)$, $B_0(r)$ assure that, for small black holes, the solution is approximated near the horizon by the Schwarzschild-Tangerlini metric in D -dimensions. The background terms $J(r,z)$, $R(r,z)$ are used to implement the topology change between event horizon and spatial infinity. A detailed discussion of the choice (A.9) is presented in [33]. Here we only note that $R(r,\pi/2) = 0$, for $r \leq 0$ (thus also for $r = r_0$), and $R(r,\pi/2) > 0$ elsewhere, which shows that $z = \pi/2$ with $r < 0$ represents the symmetry axis, while the topology of the event horizon is S^{D-2} . Also, $J(r,z)$ diverges at $r = 0, z = \pi/2$, which shows that $e^{-2B} \rightarrow 0$ there. However, this is just a coordinate singularity, as found *e.g.* by computing the Kretschmann scalar of the solutions.

The numerics is performed with the functions $\bar{A}, \bar{B}, \bar{C}$ which are finite everywhere. The equations satisfied by these functions are derived again from the Einstein equations $G_t^t = 0$, $G_r^r + G_z^z = 0$ and $G_\theta^\theta = 0$. As in the NUBS case, the remaining Einstein equations

of a black ring is $S^{D-3} \times S^1$ (*i.e.* it looks like a black string in the horizon region), the \mathcal{M}^D background being approached in the far field [57]. In the vacuum case, the only exact solution here corresponds to the $D = 5$ black ring in [58]. However, some of the black objects with a nonspherical topology of the horizon can be studied in a nonperturbative approach by employing a suitable version of the Weyl formalism [59], [60]. This leads to a much simpler numerical problem as compared to the caged BHs in KK theory.

$G_z^r = 0$, $G_r^r - G_z^z = 0$ yield two constraints, which are automatically satisfied for the set of boundary conditions chosen.

For the metric ansatz (A.7), the caged BH solutions are constructed subject to the following set of boundary conditions

$$\begin{aligned}\bar{A}|_{r=\infty} &= \bar{B}|_{r=\infty} = \bar{C}|_{r=\infty} = 0, \\ \partial_r \bar{A}|_{r=r_0} &= 0, \quad \partial_r \bar{B}|_{r=r_0} = 1 - \frac{\partial_r J}{2J}|_{r=r_0}, \quad \partial_r \bar{C}|_{r=r_0} = 1 - \frac{\partial_r R}{R}|_{r=r_0}, \\ \partial_z \bar{A}|_{z=0,\pi/2} &= \partial_z \bar{B}|_{z=0,\pi/2} = \partial_z \bar{C}|_{z=0,\pi/2} = 0.\end{aligned}\quad (\text{A.10})$$

The regularity of the solutions requires also $\bar{B} = \bar{C}$ for $r \leq 0, z = \pi/2$.

The Hawking temperature and entropy of the BH solutions are given by²³

$$T_H = \frac{(D-3)e^{A(r_0,z)-B(r_0,z)}}{4\pi\sqrt{J(r_0,z)}}, \quad S = \frac{\Omega_{D-3}}{4G} \int_0^\pi dz e^{B(r_0,z)+(D-3)C(r_0,z)} J(r_0,z) R^{D-3}(r_0,z). \quad (\text{A.11})$$

The mass and tension of the solutions can be read in principle from the asymptotics of the metric functions \bar{A}, \bar{B} (in practice, these quantities are computed usually from the event horizon quantities, by using the Smarr relation (4.3) together with the first law of thermodynamics (4.4)).

The numerical iteration starts with large values of $|r_0|$, *i.e.* small black holes, which are well approximated by the Schwarzschild-Tangerlini solution in D -dimensions. These configurations are used as initial data for the solutions with a smaller absolute value of r_0 . This results in a branch of caged black holes, which stops to exist for a critical value of $r_0 < 0$.

We have constructed in this way a number of $D = 6$ solutions, with relatively small values of r_0 . In our approach, the numerical calculations were based on the Newton-Raphson method and are performed with help of the program FIDISOL [62], which provides also an error estimate for each unknown function. The numerical accuracy of the BHs was always lower than in the NUBS case, the errors increasing around the point $r = 0, z = \pi/2$. However, the results for the solutions we have constructed so far are in good agreement with those in [35], [33]. Typical profiles of the metric functions A, B, C are shown in Figure 4 (left).

B. Slowly rotating, charged uniform black strings in Einstein-Maxwell theory

For the parametrization considered in this paper, the uniform, charged black string solution in the EM theory is given by

$$ds_0^2 = \Omega^{\frac{2}{D-3}}(r) \left(\frac{dr^2}{f(r)} + dz^2 + r^2(d\theta + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\omega_{D-5}^2) \right) - \frac{f(r)}{\Omega^2(r)} dt^2, \quad \mathcal{A} = V(r)dt, \quad (\text{B.1})$$

where

$$\Omega(r) = 1 + \left(\frac{r_0}{r} \right)^{D-4} \frac{U^2}{1-U^2}, \quad f(r) = 1 - \left(\frac{r_0}{r} \right)^{D-4}, \quad V(r) = \sqrt{\frac{2(D-2)}{(D-3)}} \frac{U}{\Omega(r)} f(r), \quad (\text{B.2})$$

²³Note that the Einstein equation $G_z^r = 0$ assures that $e^{A(r,z)-B(r,z)}/\sqrt{J(r,z)}$ is constant on the horizon.

$r_0 > 0$ being the event horizon radius and $0 \leq U < 1$.

We consider a slowly rotating ansatz which follows from (6.2):

$$ds^2 = ds_0^2 - 2a \sin^2 \theta j(r) dt d\varphi, \quad \mathcal{A} = V(r) dt + ah(r) \sin^2 \theta d\varphi,$$

where a is an infinitesimal parameter. The functions $h(r), j(r)$ are solutions of a system of second order ordinary differential equations which are found from (6.5) by replacing there $A = -\log \Omega$, $B = \log \Omega/(D-3)$, $C = \log \Omega/(D-3)$.

These equations admit the following exact solution

$$h(r) = -\frac{Q_e}{r^{D-4}\Omega(r)}, \quad (\text{B.3})$$

$$\begin{aligned} j(r) = & \Omega(r)^{\frac{2}{D-3}} \frac{Q_e}{r^{D-4}} \left(\frac{\sqrt{2}(D-4)}{U\sqrt{(D-3)(D-2)}} {}_2F_1 \left(\frac{D-2}{D-4}, \frac{3D-7}{D-3}, \frac{2(D-3)}{D-4}, \frac{U^2}{U^2-1} \left(\frac{r_0}{r} \right)^{D-4} \right) \right. \\ & \left. + \frac{(D-4)(D-1)U^2 - (D-5)(D-2)}{\sqrt{2(D-2)(D-3)}U} {}_2F_1 \left(\frac{D-2}{D-4}, \frac{2D-4}{D-3}, \frac{2(D-3)}{D-4}, \frac{U^2}{U^2-1} \left(\frac{r_0}{r} \right)^{D-4} \right) \right), \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function. An unexpected feature here is the occurrence for $D = 5$ of a logarithmic term in the expression of $J(r)$,

$$j(r) = \frac{Q_e}{\sqrt{3}U} \frac{1}{r\Omega(r)} \left(1 - \frac{1-U^2}{r_0} r\Omega(r) - \frac{2(1-U^2)^2 r^2}{r_0^2 U^2} r^2 \Omega^2(r) + \frac{2(1-U^2)^3}{r_0^3 U^4} r^3 \Omega^3(r) \log \Omega(r) \right). \quad (\text{B.4})$$

The angular momentum and magnetic dipole moment are read off from the asymptotic form of $j(r)$ and $h(r)$ (see (6.7), (6.8))

$$j(r) = \frac{Q_e(6 + 4U^2 + D(D-5)(1+U^2))}{\sqrt{2(D-2)(D-3)(D-2)}U} \frac{1}{r^{D-4}} + \dots, \quad h(r) = -\frac{Q_e}{r^{D-4}} + \dots.$$

The expression of the gyromagnetic ratio (6.10) derived in this way is (here we use also (5.29) for the mass to charge ratio):

$$g = D-3 - \frac{2U^2}{(D-2)(D-3)+(D-1)(D-4)U^2}, \quad (\text{B.5})$$

which varies in the range

$$D-3 < g < \frac{(D-2)^2(D-4)}{D(D-5)+5}. \quad (\text{B.6})$$

Thus, different from the corresponding case of (-asymptotically Minkowski) black holes in D -dimensions, the gyromagnetic ratio of the slowly rotating black strings in EM theory is not constant even in the first order of the perturbation theory. This can be understood as follows. After performing a KK reduction with respect to the Killing vector $\partial/\partial z$, the static EM uniform black strings (B.1) become black hole solutions in a Einstein-Maxwell-dilaton theory in $D-1$ dimensions, with a value $a = 1/\sqrt{2(D-2)(D-3)}$ of the dilaton coupling constant. However, the results in [63] show that the inclusion of a dilaton field

in the action modifies the value of g for the slowly rotating black holes in d -dimensions, which deviates from the standard result $d - 2$. This explains the relation (B.5) we have found for charged black strings in D -dimensions.

We close this part by remarking that a similar study can be performed for slowly rotating charged solutions with all angular momenta degrees of freedom excited. The results we have found for a single angular momentum here (and in Section 6) are expected to hold there also. In the first order of perturbation theory, all gyromagnetic ratio would be equal and given by the relation (B.5) in the UBS limit.

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